

A Brooks-type result for sparse critical graphs

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Abstract

A graph G is k -critical if it has chromatic number k , but every proper subgraph of G is $(k-1)$ -colorable. Let $f_k(n)$ denote the minimum number of edges in an n -vertex k -critical graph. Recently the authors gave a lower bound, $f_k(n) \geq \left\lceil \frac{(k+1)(k-2)|V(G)| - k(k-3)}{2(k-1)} \right\rceil$, that solves a conjecture by Gallai from 1963 and is sharp for every $n \equiv 1 \pmod{k-1}$. It is also sharp for $k=4$ and every $n \geq 6$. In this paper we refine the result by describing all n -vertex k -critical graphs G with $|E(G)| = \frac{(k+1)(k-2)|V(G)| - k(k-3)}{2(k-1)}$. In particular, this result implies exact values of $f_5(n)$ when $n \geq 7$.

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1 Introduction

A proper k -coloring, or simply k -coloring, of a graph $G = (V, E)$ is a function $f : V \rightarrow \{1, 2, \dots, k\}$ such that for each $uv \in E$, $f(u) \neq f(v)$. A graph G is k -colorable if there exists a k -coloring of G . The chromatic number, $\chi(G)$, of a graph G is the smallest k such that G is k -colorable. A graph G is k -chromatic if $\chi(G) = k$.

A graph G is k -critical if G is not $(k-1)$ -colorable, but every proper subgraph of G is $(k-1)$ -colorable. Critical graphs were first defined and used by Dirac [7, 8, 9] in 1951-52. A reason to study k -critical graphs is that every k -chromatic graph contains a k -critical subgraph and k -critical graphs have more restricted structure. For example, k -critical graphs are 2-connected and $(k-1)$ -edge-connected.

One of the basic questions on k -critical graphs is: What is the minimum number $f_k(n)$ of edges in a k -critical graph with n vertices? This question was first asked by Dirac [12] in 1957 and then was reiterated by Gallai [17] in 1963, Ore [29] in 1967 and others [21, 22, 34]. Gallai [17] has found the values of $f_k(n)$ for $n \leq 2k-1$.

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Theorem 1 (Gallai [17]) *If $k \geq 4$ and $k + 2 \leq n \leq 2k - 1$, then*

$$f_k(n) = \frac{1}{2} ((k-1)n + (n-k)(2k-n)) - 1.$$

Kostochka and Stiebitz [24] found the value $f_k(2k) = k^2 - 3$. Gallai [16] also conjectured the exact value for $f_k(n)$ for $n \equiv 1 \pmod{k-1}$.

Conjecture 2 (Gallai [16]) *If $k \geq 4$ and $n \equiv 1 \pmod{k-1}$, then*

$$f_k(n) = \frac{(k+1)(k-2)n - k(k-3)}{2(k-1)}.$$

The upper bound on $f_k(n)$ follows from Gallai's construction of k -critical graphs with only one vertex of degree at least k . So the main difficulty of the conjecture is in proving the lower bound on f_k .

For a graph G and vertex $u \in V(G)$, a *split* of u is a construction of a new graph G' such that $V(G') = V(G) - u + \{u', u''\}$, where $G - u \cong G' - \{u', u''\}$, $N(u') \cup N(u'') = N(u)$, and $N(u') \cap N(u'') = \emptyset$. A *DHGO-composition* $O(G_1, G_2)$ of graphs G_1 and G_2 is a graph obtained as follows: delete some edge xy from G_1 , split some vertex z of G_2 into two vertices z_1 and z_2 of positive degree, and identify x with z_1 and y with z_2 . Note that DHGO-composition could be found in paper by Dirac [13] and has roots in [10]. It was also used by Gallai [16] and Hajós [19]. Ore [29] used it for a composition of complete graphs.

The mentioned authors observed that if G_1 and G_2 are k -critical and G_2 is not k -critical after z has been split, then $O(G_1, G_2)$ also is k -critical. This observation implies

$$f_k(n+k-1) \leq f_k(n) + \frac{(k+1)(k-2)}{2} = f_k(n) + (k-1) \frac{(k+1)(k-2)}{2(k-1)}. \quad (1)$$

Ore believed that using this construction starting from an extremal graph on at most $2k$ vertices repeatedly with $G_2 = K_k$ at each iteration is best possible for constructing sparse critical graphs.

Conjecture 3 (Ore [29]) *If $k \geq 4$, $n \geq k$ and $n \neq k+1$, then $f_k(n+k-1) = f_k(n) + (k-2)(k+1)/2$.*

Note that Conjecture 2 is equivalent to the case $n \equiv 1 \pmod{k-1}$ of Conjecture 3.

Some lower bounds on $f_k(n)$ were obtained in [12, 28, 16, 24, 25, 15]. Recently, the authors [26] proved Conjecture 2 valid.

Theorem 4 ([26]) *If $k \geq 4$ and G is k -critical, then $|E(G)| \geq \left\lceil \frac{(k+1)(k-2)|V(G)| - k(k-3)}{2(k-1)} \right\rceil$. In other words, if $k \geq 4$ and $n \geq k$, $n \neq k+1$, then*

$$f_k(n) \geq F(k, n) := \left\lceil \frac{(k+1)(k-2)n - k(k-3)}{2(k-1)} \right\rceil.$$

The result also confirms Conjecture 3 in several cases.

Corollary 5 ([26]) *Conjecture 3 is true if (i) $k = 4$, (ii) $k = 5$ and $n \equiv 2 \pmod{4}$, or (iii) $n \equiv 1 \pmod{k-1}$.*

Some applications of Theorem 4 are given in [26] and [5]. In [27], the authors derive from a partial case of Theorem 4 a half-page proof of the well-known Grötzsch Theorem [18] that every planar triangle-free graph is 3-colorable. Conjecture 3 is still open in general. By examining known values of $f_k(n)$ when $n \leq 2k$, it follows that $f_k(n) - F(k, n) \leq k^2/8$.

The goal of this paper is to describe the k -extremal graphs, i.e. the k -critical graphs G such that $|E(G)| = \frac{(k+1)(k-2)|V(G)| - k(k-3)}{2(k-1)}$. This is a refinement of Conjecture 2: For $n \equiv 1 \pmod{k-1}$, we describe all n -vertex k -critical graphs G with $|E(G)| = f_k(n)$. This is also the next step towards the full solution of Conjecture 3.

By definition, if G is k -extremal, then $\frac{(k+1)(k-2)|V(G)| - k(k-3)}{2(k-1)}$ is an integer, and so $|V(G)| \equiv 1 \pmod{k-1}$. For example, K_k is k -extremal.

Suppose that G_1 and G_2 are k -extremal and $G = O(G_1, G_2)$. Then

$$\begin{aligned} |E(G)| &= |E(G_1)| + |E(G_2)| - 1 = \frac{(k+1)(k-2)(|V(G_1)| + |V(G_2)|) - 2k(k-3)}{2(k-1)} - 1 \\ &= \frac{(k+1)(k-2)|V(G)| - k(k-3)}{2(k-1)}. \end{aligned}$$

After z is split, G_2 will still have $F(k, |V(G_2)|) < F(k, |V(G_2)| + 1)$ edges, and therefore will not be k -critical. Thus the DHGO-composition of any two k -extremal graphs is again k -extremal.

A graph is a k -Ore graph if it is obtained from a set of copies of K_k by a sequence of DHGO-compositions. By the above, every k -Ore graph is k -extremal. So, we have an explicit construction of infinitely many k -extremal graphs.

The main result of the present paper is the following.

Theorem 6 *Let $k \geq 4$ and G be a k -critical graph. Then G is k -extremal if and only if it is a k -Ore graph. Moreover, if G is not a k -Ore graph, then $|E(G)| \geq \frac{(k+1)(k-2)|V(G)| - y_k}{2(k-1)}$, where $y_k = \max\{2k - 6, k^2 - 5k + 2\}$. Thus $y_4 = 2$, $y_5 = 4$, and $y_k = k^2 - 5k + 2$ for $k \geq 6$.*

The message of Theorem 6 is that although for every $k \geq 4$ there are infinitely many k -extremal graphs, they all have a simple structure. In particular, every k -extremal graph distinct from K_k has a separating set of size 2. The theorem gives a slightly better approximation for $f_k(n)$ and adds new cases for which we now know the exact values of $f_k(n)$:

Corollary 7 *Conjecture 3 holds and the value of $f_k(n)$ is known if (i) $k \in \{4, 5\}$, (ii) $k = 6$ and $n \equiv 0 \pmod{5}$, (iii) $k = 6$ and $n \equiv 2 \pmod{5}$, (iv) $k = 7$ and $n \equiv 2 \pmod{6}$, or (v) $k \geq 4$ and $n \equiv 1 \pmod{k-1}$.*

This value of y_k in Theorem 6 is best possible in the sense that for every $k \geq 4$, there exist infinitely many 3-connected graphs G with $|E(G)| = \frac{(k+1)(k-2)|V(G)| - y_k}{2(k-1)}$. The idea of this construction (Construction 55) and the examples for $k = 4, 5$ are due to Toft ([33], based on [32]). Construction 57 produces the examples for $k \geq 6$.

Theorem 6 has already found interesting applications. In [3], it was used to describe the 4-critical planar graphs with exactly 4 triangles. This problem was studied by Axenov [1] in the seventies, and then mentioned by Steinberg [31] (quoting Erdős from 1990), and Borodin [2]. It was proved in [3] that the 4-critical planar graphs with exactly 4 triangles and no 4-faces are exactly the 4-Ore graphs with exactly 4 triangles. Also, Kierstead and Rabern [23] and independently Postle [30]

have used Theorem 6 to describe the infinite family of 4-critical graphs G with the property that for each edge $xy \in E(G)$, $d(x) + d(y) \leq 7$. It turned out that such graphs form a subfamily of the family of 4-Ore graphs.

Our proofs will use the language of *potentials*.

Definition 8 *Let G be a graph. For $R \subseteq V(G)$, define the k -potential of R to be*

$$\rho_{k,G}(R) = (k+1)(k-2)|R| - 2(k-1)|E(G[R])|. \quad (2)$$

When there is no chance for confusion, we will use $\rho_k(R)$. Let $P_k(G) = \min_{\emptyset \neq R \subseteq V(G)} \rho_k(R)$.

Informally, $\rho_{k,G}(R)$ measures how many edges are needed to be added to $G[R]$ (or removed, if the potential is negative) in order for the resulting graph to have average degree $\frac{(k+1)(k-2)}{k-1}$. Our proofs below will involve adding and deleting edges and vertices, so using the language of potentials helps keep track of whether or not the manipulations of the graph maintain the assumptions of the theorem. By definition, adding an edge or gluing vertices together decreases the potential, and deleting edges or splitting a vertex increases the potential.

We will also use the related parameter $\tilde{P}_k(G)$ which is the minimum of $\rho_{k,G}(W)$ over all $W \subset V(G)$ with $2 \leq |W| \leq |V(G)| - 1$.

Translated into the language of potentials, Theorem 4 sounds as follows.

Corollary 9 ([26]) *If G is k -critical then $\rho_k(V(G)) \leq k(k-3)$. In particular, if $\rho_{k,G}(S) > k(k-3)$ for all nonempty $S \subseteq V(G)$, then G is $(k-1)$ -colorable.*

Similarly, our main result, Theorem 6, is:

Theorem 10 *If G is k -critical and not a k -Ore graph, then*

$$\rho_k(V(G)) \leq y_k,$$

where $y_k = \max\{2k - 6, k^2 - 5k + 2\}$. In particular, if a graph H does not contain a k -Ore graph as a subgraph and $\tilde{P}_k(H) > y_k$, then H is $(k-1)$ -colorable.

Our strategy of the proof (similar to those in [4, 6, 26, 27]) is to consider a minimum counter-example G to Theorem 10 and derive a set of its properties leading to a contradiction. Quite useful claims will be that all nontrivial proper subsets of $V(G)$ have “high” potentials. Important examples of such claims are Claim 25 and Lemma 35 below. This will help us to provide $(k-1)$ -colorings of subgraphs of G with additional properties. For example, Claim 25 will imply Claim 26 that adding any edge to a subgraph H of G with $1 < |V(H)| < |V(G)|$ leaves the subgraph $(k-1)$ -colorable. Important new ingredient of the proof is the study in the next section of the properties of k -Ore graphs and their colorings. In Section 3 we prove basic properties of our minimum counter-example G , including Claim 25 mentioned above. Then in Section 4 we introduce and study properties of *clusters* – sets of vertices of degree $k-1$ in G with the same closed neighborhood. This will allow us to prove Lemma 35. Based on this lemma and its corollaries, we prove Theorem 10 in Section 5 using some variations of discharging; the cases of small k will need separate considerations. In Section 6 we discuss the sharpness of our result and in Section 7 — some algorithmic aspects of it.

2 Potentials and Ore graphs

The fact below summarizes useful properties of ρ_k and y_k following directly from the definitions or Corollary 9.

Fact 11 *For the k -potential defined by (2), we have*

1. *Potential is submodular:*

$$\rho_k(X \cap Y) + \rho_k(X \cup Y) = \rho_k(X) + \rho_k(Y) - 2(k-1)|E_G[X - Y, Y - X]|. \quad (3)$$

2. $\rho_k(V(K_1)) = (k+1)(k-2)$.

3. $\rho_k(V(K_2)) = 2(k^2 - 2k - 1)$.

4. $\rho_k(V(K_{k-1})) = 2(k-2)(k-1)$.

5. $\rho_k(V(K_k)) = k(k-3)$.

6. *If $k \geq 4$, then $\rho_k(V(K_k)) \leq \rho_k(V(K_1)) \leq \rho_k(V(K_{k-1})) \leq \rho_k(V(K_2)) \leq \rho_k(V(K_i))$ for all $3 \leq i \leq k-2$. Furthermore, if $|S| < k$ then $\rho_k(S) \geq \rho_k(V(K_1)) = (k+1)(k-2)$.*

7. *For any vertex set S , $\rho_k(S) \geq \rho_k(K_{|S|})$. In particular, if $1 \leq |S| \leq k-1$, then $\rho_k(S) \geq (k+1)(k-2)$. If $2 \leq |S| \leq k-1$, then $\rho_k(S) \geq 2(k-2)(k-1)$.*

8. $k(k-3) \leq y_k + 2k - 2 < (k+1)(k-2)$.

9. $\rho_k(A)$ is even for each k and A .

10. *If G is a graph with a spanning subgraph H such that H is k -Ore, then $\rho_{k,G}(V(G)) \leq k(k-3)$. If $H = G$, then we have equality. If H is a proper subgraph of G , then $\rho_{k,G}(V(G)) \leq y_k$.*

A common technique in constructing critical graphs (see [21, 31]) is to use *quasi-edges* and *quasi-vertices*. For $k \geq 3$, a graph G , and $x, y \in V(G)$, a k -quasi- xy -edge $Q_k(x, y)$ is a subset Q of $V(G)$ such that $x, y \in Q$ and

(Q1) $G[Q]$ has a $(k-1)$ -coloring,

(Q2) $\phi(x) \neq \phi(y)$ for every proper $(k-1)$ -coloring of $G[Q]$, and

(Q3) for any edge $e \in G[Q]$, $G[Q] - e$ has a $(k-1)$ -coloring ϕ such that $\phi(x) = \phi(y)$.

Symmetrically, a k -quasi- xy -vertex $Q'_k(x, y)$ is a subset Q of $V(G)$ such that $x, y \in Q$ and

(Q'1) $G[Q]$ has a $(k-1)$ -coloring,

(Q'2) $\phi(x) = \phi(y)$ for every proper $(k-1)$ -coloring of $G[Q]$, and

(Q'3) for any edge $e \in G[Q]$, $G[Q] - e$ has a $(k-1)$ -coloring ϕ such that $\phi(x) \neq \phi(y)$.

If G is a critical graph, then for each $e = xy \in E(G)$, graph $G - e$ is a k -quasi- xy -vertex. On the other hand, given some k -quasi-vertices and k -quasi-edges, one can construct from copies of them infinitely many k -critical graphs. In particular, the DHGO-composition can be viewed in this way.

A quasi-edge and a quasi-vertex are very related structures. For example, if $Q_k(x, z)$ is a k -quasi- xz -vertex and we construct $Q'(x, y)$ by appending a leaf y that is adjacent only to z , then $Q'(x, y)$ is a k -quasi- xy -edge. If $Q'_k(x, y)$ is a quasi- xy -edge and $N(y) = \{z\}$, then the vertex set $Q_k(x, z) = Q'_k(x, y) - y$ is a quasi- xz -vertex.

The next observation is well known and almost trivial, but we state it, because we use it often.

Fact 12 Let $k \geq 4$. If a k -critical graph G has a separating set $\{x, y\}$, then
(1) $G - \{x, y\}$ has exactly two components, say with vertex sets A' and B' ;
(2) $xy \notin E(G)$;
(3) one of $A' \cup \{x, y\}$ and $B' \cup \{x, y\}$ is a k -quasi- xy -edge and the other is a k -quasi- xy -vertex.

Fact 12 together with the definition of k -Ore graphs, implies the following.

Fact 13 Every k -Ore graph $G \neq K_k$ has a separating set $\{x, y\}$ and two vertex subsets $A = A(G, x, y)$ and $B = B(G, x, y)$ such that
(i) $A \cap B = \{x, y\}$, $A \cup B = V(G)$ and no edge of G connects $A - x - y$ with $B - x - y$,
(ii) the graph $\tilde{G}(x, y)$ obtained from $G[A]$ by adding edge xy is a k -Ore graph,
(iii) the graph $\check{G}(x, y)$ obtained from $G[B]$ by gluing x with y into a new vertex $x * y$ is a k -Ore graph, and
(iv) $xy \notin E(G)$.

In terms of Fact 13, G is the DHGO-composition of $\tilde{G}(x, y)$ and $\check{G}(x, y)$, and we will say that $\tilde{G}(x, y)$ and $\check{G}(x, y)$ are x, y -children (or simply children) of G . Moreover, $\tilde{G}(x, y)$ will be always the *first child* and $\check{G}(x, y)$ will be the *second child*. We will repeatedly use the notation in this fact. The next fact directly follows from the definitions.

Fact 14 Using the notation in Fact 13, we have

1. A is a k -quasi- xy -vertex;
2. B is a k -quasi- xy -edge;
3. $\rho_{k,G}(A) = \rho_{k,K_1}(V(K_1)) = (k+1)(k-2)$;
4. $\rho_{k,G}(B) = \rho_{k,K_2}(V(K_2)) = 2(k^2 - 2k - 1)$;
5. $N(x) \cap B \cap N(y) = \emptyset$;
6. $N_{\tilde{G}}(v) = N_G(v)$ for each $v \in A - x - y$;
7. $d_{\check{G}}(v) = d_G(v)$ for each $v \in B - x - y$;
8. If $R \subseteq V(\tilde{G} - x)$ (respectively, $R \subseteq V(\check{G} - x * y)$), then $\rho_{k,G}(R) = \rho_{k,\tilde{G}}(R)$ (respectively, $\rho_{k,G}(R) = \rho_{k,\check{G}}(R)$). A symmetric statement for $R \subseteq V(\tilde{G} - y)$ is also true.
9. $\rho_{k,G}(V(G)) = \rho_{k,V(\tilde{G})}(V(\tilde{G})) = \rho_{k,V(\check{G})}(V(\check{G})) = k(k-3)$.

Claim 15 For every k -Ore graph G and every nonempty $R \subsetneq V(G)$, we have $\rho_{k,G}(R) \geq (k+1)(k-2)$.

Proof. Let G be a smallest counter-example to the claim. If $G = K_k$, then the statement immediately follows from Fact 11. So suppose $G \neq K_k$. By Fact 13 there is a separating set $\{x, y\}$ and two vertex subsets $A = A(G, x, y)$ and $B = B(G, x, y)$ be as in Fact 13. By the minimality of G , every proper subset of $V(\tilde{G}(x, y))$ and of $V(\check{G}(x, y))$ has potential at least $(k+1)(k-2)$. Let R have the smallest size among nonempty proper subsets of $V(G)$ with connected $G[R]$ and

$\rho_{k,G}(R) < (k+1)(k-2)$. If $\rho_{k,G}(R') < (k+1)(k-2)$ and $G[R']$ is disconnected, then the vertex set of some component of $G[R']$ also has potential less than $(k+1)(k-2)$. So, such R exists. Since $\rho_{k,G}(R) < (k+1)(k-2)$ and R is non-empty, $|R| \geq k$.

Case 1: $\{x, y\} \cap R = \emptyset$. Then, since $G[R]$ is connected, R is a non-empty proper subset either of A or B . This contradicts Fact 14 and the minimality of G .

Case 2: $\{x, y\} \cap R = \{x\}$. The set $R \cap A$ induces a non-empty connected subgraph of G , and so by the minimality of $|R|$, $\rho_{k,G}(R \cap A) \geq (k+1)(k-2)$. Similarly, $\rho_{k,G}(R \cap B) \geq (k+1)(k-2)$. By submodularity,

$$\rho_{k,G}(R) = \rho_{k,G}(R \cap A) + \rho_{k,G}(R \cap B) - \rho_{k,G}(\{x\}) \geq (k+1)(k-2),$$

a contradiction.

Case 3: $\{x, y\} \subseteq R$. If $A \subseteq R$, then by Facts 11 and 14

$$\rho_{k,\tilde{G}(x,y)}((R-A) + x * y) = \rho_{k,G}(R) - \rho_{k,G}(A) + \rho_{k,\tilde{G}(x,y)}(\{x * y\}) = \rho_{k,G}(R).$$

But by the minimality of G , this is at least $(k+1)(k-2)$, a contradiction. Similarly, if $B \subseteq R$, then

$$\rho_{k,\tilde{G}(x,y)}(R \cap A) = \rho_{k,G}(R) - \rho_{k,G}(B) + \rho_{k,\tilde{G}(x,y)}(\{x, y\}) = \rho_{k,G}(R),$$

a contradiction again. So, suppose $A - R \neq \emptyset$ and $B - R \neq \emptyset$. By the minimality of G , we have $\rho_{k,\tilde{G}(x,y)}(R \cap A) \geq (k+1)(k-2)$. Since xy is an edge in $\tilde{G}(x, y)$ but not in G , this yields $\rho_{k,G}(R \cap A) \geq (k+1)(k-2) + 2(k-1)$. Similarly, $\rho_{k,\tilde{G}(x,y)}((R-A) + x * y) \geq (k+1)(k-2)$ and thus $\rho_{k,G}(R \cap B) \geq 2(k+1)(k-2)$. Then

$$\rho_{k,G}(R) = \rho_{k,G}(R \cap A) + \rho_{k,G}(R \cap B) - 2\rho_{k,G}(K_1) \geq (k+1)(k-2) + 2(k-1),$$

a contradiction. \square

A set S of vertices in a graph G is *standard*, if

- (a) $\rho_{k,G}(S) = (k+1)(k-2)$ and
- (b) G has a separating set $\{x, y\}$ such that $\{x, y\} \subset S$ and $S - \{x, y\}$ induces a component of $G - \{x, y\}$, and
- (c) S is a k -quasi- $\{x, y\}$ -vertex.

For a standard set S , the vertices x and y in the separating set $\{x, y\}$ will be called the *border vertices* of S .

Note that a standard set is a k -quasi-vertex whose k -potential is the same as that of a vertex. The next lemma shows that every proper vertex subset of G with potential equal to that of a vertex contains a standard set.

Lemma 16 *Let G be a k -Ore graph. Let $W \subset V(H)$ with $|W| \geq 2$ and $\rho_k(W) \leq (k+1)(k-2)$. Then $G[W]$ is connected and contains a standard set.*

Proof. If $\rho_{k,G}(W) \leq (k+1)(k-2)$ and $G[W]$ is disconnected, then the vertex set of some component of $G[W]$ has potential strictly less than $(k+1)(k-2)$. This, or if $\rho_k(W) < (k+1)(k-2)$, contradicts Claim 15. So the first part follows and we may assume $\rho_k(W) = (k+1)(k-2)$.

To prove the second part, choose a counter-example G with the fewest vertices and a minimum $W \subset V(G)$ with $|W| \geq 2$ and $\rho_k(W) = (k+1)(k-2)$ that does not contain a standard subset. By Fact 11, the graph K_k simply does not have sets W with $|W| \geq 2$ and $\rho_k(W) = (k+1)(k-2)$. So $G \neq K_k$ and thus by Fact 13 has a separating set $\{x, y\}$. Consider the children graphs $\tilde{G}(x, y)$ and $\check{G}(x, y)$ defined by Fact 13. First we show that

$$G[W] \text{ is 2-connected.} \quad (4)$$

Indeed, suppose not. Then by the first part of the lemma, $G[W]$ has a cut vertex, say z . Let W_1 and W_2 be two subsets of W such that $W_1 \cap W_2 = \{z\}$, $W_1 \cup W_2 = W$ and there are no edges between $W_1 - z$ and $W_2 - z$. Then by Fact 11(1), $\rho_{k,G}(W_1) + \rho_{k,G}(W_2) = \rho_{k,G}(W) + \rho_{k,G}(\{z\}) = 2(k+1)(k-2)$. So by Claim 15, $\rho_{k,G}(W_1) = \rho_{k,G}(W_2) = (k+1)(k-2)$. Thus by the minimality of W , each of W_1 and W_2 contains a standard subset, a contradiction to the choice of G and W . This proves (4).

Let $W_A = A \cap W$, $W_B = B \cap W$, and $W'_B = W_B - \{x, y\} + x * y$. Suppose $S \subseteq W_A$ (respectively $S \subseteq W_B$), $\rho_{k,G}(S) = (k+1)(k-2)$, and $y \notin S$. Because (a) by Fact 14.8 W has the same potential in \tilde{G} (respectively \check{G}) as in G , (b) by Fact 13.ii \tilde{G} (respectively \check{G}) is also k -Ore, and (c) the minimality of G ,

$$S \text{ contains a standard set } W' \text{ in } \tilde{G}(x, y). \quad (5)$$

In each of the three cases we will use

Case 1: $W \subset A$. If $\{x, y\} \subseteq W$, then $\rho_{k,\tilde{G}(x,y)}(W) = \rho_{k,G}(W) - 2(k-1) = k(k-3)$, which by Claim 15 means that $W = A$. But A is a standard set, a contradiction to the choice of W . So we may assume symmetrically that $y \notin W$. Using (5) with $S = W$, we have that W contains a standard set W' in $\tilde{G}(x, y)$. If $x \in W'$, then it is one of the two border vertices of W' because $y \notin W \supset W'$ and $xy \in E(\tilde{G})$. Because $W \subset A - y$ we have that $W' \subset W \subset V(\tilde{G} - y)$ and so by Fact 14.8 W' has the same potential in G as in \tilde{G} . W' has the same border vertices in G as in \tilde{G} by Fact 14.6 and that if $x \in W'$ it was also a border vertex in \tilde{G} . So W' is also a standard set in G with the same border vertices.

Case 2: $W \subset B$. If $\{x, y\} \subseteq W$, then $\rho_{k,\check{G}(x,y)}(W'_B) = \rho_{k,G}(W) - (k+1)(k-2) = 0$, which contradicts Claim 15. If $\{x, y\} \cap W = \emptyset$, then using (5) with $S = W$ we have that W contains a standard set W' in $\check{G}(x, y)$. Moreover W' does not contain $x * y$. As in Case 1, by Fact 14.8 W' has the same potential in G as in \check{G} and W' has the same border vertices in G as in \check{G} by Fact 14.7. So W' is also a standard set in G . Thus by the symmetry between x and y we may assume $\{x, y\} \cap W = \{x\}$.

If y has no neighbors in W , then the argument follows almost the same line. The logic behind Fact 14.8 that W will have the same potential in G as in \check{G} is still true, even though $x \in W$. So the conclusion in (5) still holds and W will contain a standard set W' who will have the same potential in G as in \check{G} . Note that y has at least one neighbor in B , and by assumption that neighbor is not in $W \supseteq W'$, so it must be that if $x * y \in W'$ only if it is a border vertex. Finally, Fact 14.7, that y has no neighbors in $W \subseteq W'$, and that if $x * y \in W'$ only if it is a border vertex, implies that W' has the same border vertices (with x replacing $x * y$ if $x * y \in W'$). So W' is a standard set in G . Thus we may assume that y has exactly $i > 0$ neighbors in W .

Note that $\rho_{k,\check{G}(x,y)}(W'_B) = \rho_{k,G}(W) - 2i(k-1) = k(k-3) - (i-1)2(k-1)$. By Claim 15 and the definition of Ore-graphs, this yields that $W'_B = V(\check{G}(x, y))$ and $i = 1$. It follows that $W = B - y$, and y has exactly one neighbor, say z in W . By the discussion directly above Fact 11 (and that B is a quasi-edge) this means that W is a k -quasi- xz -vertex, and therefore a standard set.

Case 3: $W - A \neq \emptyset$ and $W - B \neq \emptyset$. Then by (4), $\{x, y\} \subseteq W$. If $W_A = A$, then we are done, since A is standard. Suppose that $W_B = B$. Since $\rho_{k,G}(B) = 2(k^2 - 2k - 1)$ by Fact 14(4), we have

$$\begin{aligned} \rho_{k,\tilde{G}(x,y)}(W_A) &= \rho_{k,G}(W) - \rho_{k,G}(B) + \rho_{k,\tilde{G}(x,y)}(\{x, y\}) \\ &= (k+1)(k-2) - 2(k^2 - 2k - 1) + 2(k+1)(k-2) - 2(k-1) = (k+1)(k-2). \end{aligned} \quad (6)$$

So $S = W_A$ satisfies the conditions (a), (b), (c) that imply (5) - although we do not satisfy the assumptions that imply (a), (b), and (c). Still, we reach the conclusion implied by (a), (b), and (c) in that W_A contains a standard set W' in $\tilde{G}(x, y)$. If $|W' \cap \{x, y\}| \leq 1$, then W' is a standard set in G with the same border vertices using the same argument as in Case 1. Otherwise, (6) with $W', W' \cup B$ in replace of W_A, W respectively says that $\rho_{k,G}(W' \cup B) = \rho_{k,\tilde{G}(x,y)}(W') = (k+1)(k-2)$.

We claim that $W' \cup B$ has the same border vertices in G as W' does in \tilde{G} : by Fact 14.6 the only new border vertices in $W' \cap A$ could be x or y . But their only new neighbors are in B , and $B \subset W' \cup B$ so they can not be *new* border vertices. The other consideration is the vertices in $B - x - y$, but $N(B - x - y) = \{x, y\} \subset B \cup W'$ so they are not border vertices. This means that $W' \cup B \subseteq W$ is a standard set in G .

Thus the last possibility is that $W_A \neq A$, and $W_B \neq B$. By Claim 15 and that \check{G} and \tilde{G} are each k -Ore, $\rho_{k,\tilde{G}}(W_A) \geq (k+1)(k-2)$ and $\rho_{k,\check{G}}(W'_B) \geq (k+1)(k-2)$. Because $\{x, y\} \subset W_A, W_B$, it is a direct calculation that $\rho_{k,\tilde{G}}(W_A) = \rho_{k,G}(W_A) - 2(k-1)$ and $\rho_{k,\check{G}}(W'_B) = \rho_{k,G}(W_B) - (k+1)(k-2)$. Therefore

$$\rho_{k,G}(W_A) + \rho_{k,G}(W_B) \geq (k+1)(k-2) + 2(k-1) + (k+1)(k-2) + (k+1)(k-2) = 3(k+1)(k-2) + 2(k-1).$$

But since $W_A \cap W_B = \{x, y\}$ and $xy \notin E(G)$, by Fact 11(1),

$$\rho_{k,G}(W_A) + \rho_{k,G}(W_B) = \rho_{k,G}(W) + \rho_{k,G}(\{x, y\}) = 3(k+1)(k-2),$$

a contradiction. \square

Now we will prove two statements on colorings and structure of subgraphs not containing standard sets of k -Ore graphs.

Lemma 17 *Let G be a k -Ore graph. Let uv be an edge in G such that*

$$\rho_{k,G-uv}(W) > (k+1)(k-2) \text{ for every } W \subseteq V(G-uv) \text{ with } 2 \leq |W| \leq |V(G)| - 1. \quad (7)$$

Then for each $w \in V(G) - u - v$, there is a $(k-1)$ -coloring ϕ_w of $G - uv$ such that $\phi_w(w) \neq \phi_w(u) = \phi_w(v)$.

Proof. We use induction on $|V(G)|$. For $G = K_k$, the statement is evident. Otherwise, let $x, y, A, B, \tilde{G}(x, y)$ and $\check{G}(x, y)$ be as in Fact 13. By Fact 14, $\rho_{k,G}(A) = (k+1)(k-2)$, and thus our assumption that $\rho_{k,G-uv}(W) > (k+1)(k-2)$ for every $W \subseteq V(G-uv)$, we have $uv \in G[A]$.

Case A: $w \in A$. By the induction assumption, there exists a $(k-1)$ -coloring ϕ'_w of $\tilde{G}(x, y) - uv$ such that $\phi'_w(w) \neq \phi'_w(u) = \phi'_w(v)$. Since $\phi'_w(x) \neq \phi'_w(y)$ and B is a quasi- xy -edge, this coloring extends to a $(k-1)$ -coloring of the whole G .

Case B: $w \in B - x - y$. Let ϕ' be any $(k-1)$ -coloring of $\tilde{G}(x, y) - uv$. By Fact 13 $xy \notin E(G)$, so $uv \neq xy$. Fact 13 also says that $xy \in E(\tilde{G})$, so in total we have $\phi'(x) \neq \phi'(y)$. Since $\tilde{G}(x, y)$ is k -critical, $\phi'(u) = \phi'(v)$.

Case B1: $\phi'(u) = \phi'(x)$. Let $G_0 = G[B] + xw$. Note that for $W \subseteq V(G_0)$, we have $\rho_{k,G_0}(W) = \rho_{k,G}(W)$ if $\{x, w\} \not\subseteq W$ and $\rho_{k,G_0}(W) = \rho_{k,G}(W) - 2(k-1)$ if $\{x, w\} \subseteq W$. Since $u, v \notin V(G_0)$, we have by (7) that $\rho_{k,G}(W) > (k+1)(k-2)$ for any $W \subseteq V(G_0)$ with $|W| > 1$. Those two statements together imply that

$$\rho_{k,G_0}(W) \geq \rho_{k,G}(W) - 2(k-1) > (k+1)(k-2) - 2(k-1) = k(k-3)$$

for every $W \subseteq V(G_0)$ with $|W| > 1$. If $|W| = 1$, then $\rho_{k,G_0}(W) = (k+1)(k-2) > k(k-3)$, and so this bound holds for all subsets of $V(G_0)$. By the second part of Corollary 9, this implies that G_0 has a $(k-1)$ -coloring ϕ'' . Since $G[B] \subset G_0[B]$, it follows that ϕ'' is also a coloring of quasi- xy -edge $G[B]$, which means that $\phi''(x) \neq \phi''(y)$. By Fact 13(i) and because $\phi''(x) \neq \phi''(y)$, we can rename the colors in ϕ'' so that $\phi''(x) = \phi'(x)$ and $\phi''(y) = \phi'(y)$, and obtain a $(k-1)$ -coloring $\phi = \phi'|_A \cup \phi''|_B$. By construction, $\phi(u) = \phi(x) \neq \phi(w)$.

Case B2: $\phi'(u) \notin \{\phi'(x), \phi'(y)\}$ and $k \geq 5$. Take any $(k-1)$ -coloring ϕ'' of $G[B]$ such that $\phi''(x) = \phi'(x)$ and $\phi''(y) = \phi'(y)$ (B is a quasi-edge, so we can do this). If $\phi''(w) \in \{\phi''(x), \phi''(y)\}$, then by the assumption of the case $\phi = \phi'|_A \cup \phi''|_B$ is the $(k-1)$ -coloring we are looking for. Otherwise, since $k-1 \geq 4$, we can rename the colors of ϕ'' distinct from the colors of x and y so that $\phi''(w) \neq \phi'(u)$ and again take $\phi = \phi'|_A \cup \phi''|_B$.

Case B3: $\phi'(u) \notin \{\phi'(x), \phi'(y)\}$ and $k = 4$. Let G_0 be obtained from $G[B]$ by adding a new vertex z adjacent to x, y and w . Suppose first that G_0 has a 3-coloring ϕ'' . Since $G[B] \subset G_0$ and B is a quasi- xy -edge, $\phi''(x) \neq \phi''(y)$. So z has the color distinct from $\phi''(x)$ and $\phi''(y)$, and thus because there are only 3 colors, $\phi''(w) \in \{\phi''(x), \phi''(y)\}$. In this case by renaming the colors in ϕ'' so that $\phi''(x) = \phi'(x)$ and $\phi''(y) = \phi'(y)$, we get a required coloring of G . Now suppose that G_0 has no 3-coloring. Then G_0 contains a 4-critical subgraph G_1 . Since G_1 is not a subgraph of G , it follows that $z \in V(G_1)$. Since G_1 is 4-critical, $\delta(G_1) \geq 4-1 = 3$, and so $\{x, y, w\} \subset V(G_1)$. Let $W = V(G_1)$. Since $\rho_{4,G_0}(W) \leq 4$ by Corollary 9, we have $\rho_{4,G}(W-z) = \rho_{4,G}(W) - 10 + 3(6) \leq 12$. So Fact 11(1) (because $G[A \cap W] = G[\{x, y\}] \cong 2K_1$) implies that ,

$$\rho_{4,G}(A \cup W - z) \leq \rho_{4,G}(A) + \rho_{4,G}(W - z) - 2\rho_4(K_1) \leq 10 + 12 - 20 = 2.$$

By Claim 15, this yields that $A \cup W - z$ either is empty or is $V(G)$. But $A \cup W - z \neq V(G)$, since G is 4-Ore, and the vertex set of each 4-Ore graph has potential $k(k-3) = 4$ by Fact 11. Also $|A \cup W - z| \geq 3$ because $\{x, y, w\} \subset W - z$. \square

Claim 18 *Let G be a k -Ore graph. Let u be a vertex in G such that*

$$\rho_{k,G}(W) > (k+1)(k-2) \text{ for every } W \subseteq V(G) - u \text{ with } |W| \geq 2. \quad (8)$$

Then there exists a $(k-1)$ -clique S such that $d_G(v) = k-1$ for all $v \in S$ and $(N(S) - S)$ is an independent set.

Proof. We use induction on $|V(G)|$. For $G = K_k$, the statement is evident. Otherwise, let $x, y, A, B, \tilde{G}(x, y)$ and $\check{G}(x, y)$ be as in Fact 13. Then $\rho_{k,G}(A) = (k+1)(k-2)$, and so $u \in A$.

If there exists a $W \subseteq V(\tilde{G}(x, y))$ such that $|W| \geq 2$ and $\rho_{k,\tilde{G}(x,y)}(W) \leq (k+1)(k-2)$, then by (8), $x*y \in W$. So by induction, \tilde{G} has a set $S \subseteq V(\tilde{G}(x, y)) - x*y$ such that $\tilde{G}(x, y)[S] \cong K_{k-1}$, $d_{\tilde{G}(x,y)}(v) = k-1$ for all $v \in S$, and $(N(S) - S)$ is an independent set in $\tilde{G}(x, y)$. Recall that $\check{G} - x*y$ is a subgraph of G , and since $i \in A$ we have $S \subseteq V(G) - u$, $G[S] \cong K_{k-1}$, and $(N_G(S) - S)$ is an independent set in G . By Fact 14(7), $d_G(v) = k-1$ for all $v \in S$. \square

3 Basic properties of minimal counter-examples

The *closed neighborhood* of a vertex u in a graph H is $N_H[u] = N_H(u) \cup \{u\}$. We will use the following partial order on the set of graphs. A graph H is *smaller than* a graph G , if either

- (S1) $|V(G)| > |V(H)|$, or
- (S2) $|V(G)| = |V(H)|$ and $|E(G)| > |E(H)|$, or
- (S3) $|V(G)| = |V(H)|$, $|E(G)| = |E(H)|$ and G has fewer pairs of adjacent vertices with the same closed neighborhood.

Note that if H is a subgraph of G , then H is smaller than G . Let $k \geq 4$ and G be a minimal with respect to relation “smaller” counter-example to Theorem 10: G is a k -critical graph with $\rho_k(V(G)) > y_k$ that is not k -Ore. Let $n := |V(G)|$. In this section, we derive basic properties of G and its colorings.

Claim 19 G is 3-connected.

Proof. Suppose that G has a separating set $\{x, y\}$ and sets $A \subset V(G)$ and $B \subset V(G)$ such that $A \cap B = \{x, y\}$, $A \cup B = V(G)$, and no edge of G connects $A - x - y$ with $B - x - y$. By Fact 12 and the symmetry between A and B , we may assume that A is a k -quasi- xy -vertex and B is a k -quasi- xy -edge. It follows that the graph \tilde{G} obtained from $G[A]$ by inserting edge xy and the graph \check{G} obtained from $G[B]$ by gluing x with y are k -critical. Then

$$\begin{aligned} \rho_k(V(G)) &\leq (\rho_k(V(\tilde{G})) + 2(k-1)) + (\rho_k(V(\check{G})) + (k+1)(k-2)) - 2 \cdot (k+1)(k-2) \\ &= \rho_k(V(\tilde{G})) + \rho_k(V(\check{G})) - k(k-3). \end{aligned}$$

By assumption, $y_k < \rho_k(V(G))$. By Corollary 9, $\rho_{k,\tilde{G}}(V(\tilde{G})) \leq k(k-3)$ and $\rho_k(V(\check{G})) \leq k(k-3)$. Moreover, if \tilde{G} (respectively, \check{G}) is not a k -Ore graph, then by minimality of G , the potential of its vertex set is at most y_k . If at least one of \tilde{G} or \check{G} is not k -Ore, then we get a contradiction. If both are k -Ore, then G is k -Ore, which contradicts the definition of G . \square

Fact 20 By the definition of ρ_k and the assumption $\rho_k(V(G)) > y_k$, for each $v \in V(G)$,

$$\rho_k(V(G) - v) = \rho_k(V(G)) - (k+1)(k-2) + 2(k-1)d(v) >$$

- $y_k + k^2 - 3k + 4$, if $d(v) = k-1$,
- $y_k + k^2 - k + 2$, if $d(v) = k$,
- $y_k + k^2 + k$, if $d(v) \geq k+1$.

Because $y_k \geq k^2 - 5k + 2$, we see that $\rho_k(V(G) - v)$ is also more than

- $2k^2 - 8k + 6 = 2(k-3)(k-1)$, if $d(v) = k-1$,
- $2k^2 - 6k + 4 = 2(k-2)(k-1)$, if $d(v) = k$,
- $2k^2 - 4k + 2 = 2(k-1)^2$, if $d(v) \geq k+1$.

Now we define graph $Y(G, R, \phi)$. The idea of $Y(G, R, \phi)$ is that it is often smaller than G , and every $(k-1)$ -coloring of it extends to a $(k-1)$ -coloring of G .

Definition 21 For a graph G , a set $R \subset V(G)$ and a $(k-1)$ -coloring $\phi : R \rightarrow [k-1]$ of $G[R]$, the graph $Y(G, R, \phi)$ is constructed as follows. Let $R_* = \{v \in R : N(v) - R \neq \emptyset\}$. Let t be the number of colors used on R_* . We may renumber the colors so that the colors used on R_* are $1, \dots, t$. First, for $i = 1, \dots, t$, let R'_i denote the set of vertices in $V(G) - R$ adjacent in G to at least one vertex $v \in R$ with $\phi(v) = i$. Now, let $Y(G, R, \phi)$ be obtained from $G - R$ by adding a set $X = \{x_1, \dots, x_t\}$ of new vertices such that $N(x_i) = R'_i \cup (\{x_1, \dots, x_t\} - x_i)$ for $i = 1, \dots, t$.

Informally, the definition can be rephrased as follows: For a given $R \subset V(G)$ and a $(k-1)$ -coloring ϕ of $G[R]$, we glue each color class of $\phi(G[R])$ into a single vertex, then add all possible edges between the new vertices (corresponding to the color classes) and then delete those that have no neighbors outside of R . $Y(G, R, \phi)$ will be a useful gadget for deriving properties of G , since it inherits a lot of structure from G .

First we will prove some useful properties of $Y(G, R, \phi)$.

Claim 22 Suppose $R \subset V(G)$ and ϕ is a $(k-1)$ -coloring of $G[R]$. Then $\chi(Y(G, R, \phi)) \geq k$.

Proof. Let $G' = Y(G, R, \phi)$. Suppose G' has a $(k-1)$ -coloring ϕ' . By the construction of G' , the colors of all x_i in ϕ' are distinct. We can change the names of the colors so that $\phi'(x_i) = i$ for $1 \leq i \leq t$, where t is given in Definition 21. By the construction of G' , $\phi'(u) \neq i$ for each vertex $u \in R'_i$. Therefore $\phi|_R \cup \phi'|_{V(G)-R}$ is a proper coloring of G , a contradiction. \square

The next statement is a submodularity-type equation that is a direct extension of Fact 11(1).

Claim 23 Let $R \subset V(G)$, ϕ be a $(k-1)$ -coloring of $G[R]$ and $G' = Y(G, R, \phi)$. Let $W \subseteq V(G')$. If $W \cap X = \{x_{i_1}, \dots, x_{i_q}\}$, then let $R|_W$ denote the set of vertices $v \in R_*$ such that $\phi(v) \in \{i_1, \dots, i_q\}$. Then

$$\rho_{k,G}(W - X + R) = \rho_{k,G'}(W) - \rho_{k,G'}(W \cap X) + \rho_{k,G}(R) - 2(k-1)|E_G(W - X, R - R|_W)|. \quad (9)$$

Proof. Since $\rho_{k,G}(U)$ is a linear combination of the numbers of vertices and edges in $G[U]$, it is enough to check that every vertex and edge of $G[W - X + R]$ is accounted exactly once in the RHS of (9) and the weight of every other vertex or edge either does not appear at all or appears once with plus and once with minus. In particular, the weight of every vertex and edge of $G'[W \cap X]$ appears once with plus and once with minus. \square

By Corollary 9 and Claim 22, $Y(G, R, \phi)$ contains a vertex set with potential at most $k(k-3)$. In some instances this will not be enough for our purposes, and we will want $Y(G, R, \phi)$ to contain a vertex set with potential at most y_k . The next claim helps us with this.

Claim 24 For any $R \subset V(G)$ with proper $(k-1)$ -coloring ϕ of $G[R]$, let $Y = Y(G, R, \phi)$. Then there exists an $S \subseteq V(Y)$ that is spanned by a k -critical graph, and so $\rho_{k,Y}(S) \leq k(k-3)$. Furthermore, if $|R| \geq k$, then $Y(G, R, \phi)$ is smaller than G and

(a) Y contains a k -Ore subgraph with vertex set S , or

(b) we have the stronger bound $\rho_{k,Y}(S) \leq y_k$.

Moreover, $S \cap X \neq \emptyset$.

Proof. That Y has some k -critical subgraph F' follows from Claim 22. The bound on the potential of $S = V(F')$ follows from Corollary 9. In order to prove the “Furthermore” part, observe that if $|R| \geq k$, then $Y(G, R, \phi)$ is smaller than G by Rule (S1) in the definition of “smaller”, since ϕ uses at most $k - 1 < |R|$ colors on R . By subgraphs, F' is smaller than Y , and so by transitivity F' is smaller than Y . So, (a) or (b) holds by the minimality of G , with $S = V(F')$. The last part comes from the fact that G is critical. \square

Now we will use $Y(G, R, \phi)$ to prove lower bounds on potentials of nontrivial sets.

Claim 25 *If $\emptyset \neq R \subsetneq V(G)$, then $\rho_{k,G}(R) \geq \rho_k(V(G)) + 2(k - 1) > y_k + 2(k - 1)$.*

Proof. Let R have the smallest potential among nonempty proper subsets of $V(G)$. Since G is k -critical, $G[R]$ has a proper coloring $\phi : R \rightarrow [k - 1]$. Let $G' = Y(G, R, \phi)$, X be as in Definition 21. By Claim 24, G' contains a subset S with potential at most $k(k - 3)$ and $S \cap X \neq \emptyset$. Let $Z = S - X + R$. Because $|X| \leq k - 1$, by Fact 11 each non-empty subgraph of X has potential at least $(k + 1)(k - 2)$. So by (9),

$$\begin{aligned} \rho_{k,G}(Z) &\leq \rho_{k,G'}(S) - \rho_{k,G'}(S \cap X) + \rho_{k,G}(R) \\ &\leq k(k - 3) - (k + 1)(k - 2) + \rho_{k,G}(R) = \rho_{k,G}(R) - 2(k - 1). \end{aligned} \tag{10}$$

Since $Z \supset R$, it is nonempty. So, by the minimality of the potential of R , we have $Z = V(G)$.

The final statement comes from our assumption that $\rho_k(V(G)) > y_k$. \square

By Claim 19, $V(G)$ can not be partitioned into a k -quasi-edge and a k -quasi-vertex. The following is a strengthening of the fact this fact: it implies that G has no quasi-vertex.

Claim 26 *For each $R \subsetneq V(G)$ with $|R| \geq 2$ and any distinct $x, y \in R$, the graph $G[R] + xy$ is $(k - 1)$ -colorable.*

Proof. Let R be a smallest subset of vertices such that $2 \leq |R| < n$ and for some distinct $xy \in R$, the graph $H = G[R] + xy$ is not $(k - 1)$ -colorable. Since G is k -critical, $xy \notin E(G)$. By the minimality of R , graph H is vertex-critical - and thus any (edge-)critical subgraph of H has vertex set R .

By Claim 25, $\rho_{k,H}(R) = -(2k - 2) + \rho_{k,G}(R) \geq k(k - 3)$. Because $|R| < n$, by Rule (S1) H is smaller than G . By the minimality of G and because $k(k - 3) > y_k$, any k -critical subgraph of H must be k -Ore. In summary, H contains a k -Ore spanning subgraph H_1 . By Fact 11, $\rho_{k,H_1}(R) = k(k - 3)$.

If $H_1 \neq H$, then $H[R]$ has at least one more edge than $H_1[R]$. But $H[R]$ has just one more edge than $G[R]$, so this would mean that $\rho_{k,G}(R) \leq \rho_{k,H_1}(R) = k(k - 3)$, a contradiction to Claim 25. Hence, $H = H_1$, and thus H is a k -Ore graph by itself. Moreover,

$$\rho_{k,G}(R) = k(k - 3) + 2(k - 1) = (k + 1)(k - 2). \tag{11}$$

Recall that R_* is the set of vertices in R that have a neighbor outside of R . By Claim 19, $|R_*| \geq 3$. We want to prove that

$$G[R] \text{ has a } (k - 1)\text{-coloring } \psi \text{ such that } R_* \text{ is not monochromatic.} \tag{12}$$

Case 1: $\{x, y\} \subset R_*$. Since $|R_*| \geq 3$, we may choose $w \in R_* - x - y$. If there exists a subset $R' \subsetneq R$ with $|R'| \geq 2$ such that $\{x, y\} \not\subset R'$ and $\rho_k(R') = (k+1)(k-2)$, then by Lemma 16, H contains a standard set $A \subseteq R'$. But then there exists a pair of vertices $\{a, b\} \subset A \subseteq R' \subsetneq R$ such that $G[A] + ab$ is not $(k-1)$ -colorable, which contradicts the minimality of R . By Lemma 17, there is a $(k-1)$ -coloring ϕ_w of $H - xy$ such that $\phi_w(w) \neq \phi_w(x) = \phi_w(y)$. Then for $\psi = \phi_w$, (12) holds.

Case 2: $\{x, y\} \not\subset R_*$. Let u, v be any vertices in R_* . If $uv \in E(G)$, then (12) is immediately true. Otherwise, let $H_0 = G[R] + uv$. If H_0 has a $(k-1)$ -coloring, then (12) holds. If not, then by the minimality of R , exactly as above, H_0 is a k -Ore graph. So, we have Case 1. This proves (12).

Let ψ satisfy (12). Let $G' = Y(G, R, \psi)$ and X be as in Definition 21 of $Y(G, R, \psi)$. By Claim 24, G' contains a vertex set W such that $\rho_{k,G'}(W) \leq k(k-3)$ and $W \cap X \neq \emptyset$. Recall that X is a copy of a subgraph of K_{k-1} and that from Fact 11 the subgraph of K_{k-1} with smallest potential is $\rho_k(V(K_1)) = (k+1)(k-2)$ and the subgraph with second smallest potential is $\rho_k(V(K_{k-1})) = 2(k-2)(k-1)$. This together with (11) and the choice of W yields

$$\rho_{k,G}(W - X + R) \leq \rho_{k,G'}(W) - \rho_{k,G'}(X \cap W) + \rho_{k,G}(R) \leq k(k-3) - (k+1)(k-2) + (k+1)(k-2) = k(k-3). \quad (13)$$

Since $W - X + R \supset R$, we have $|W - X + R| \geq 2$. From Fact 11 $y_k + (2k-2) \geq k(k-3)$, and when combined with Claim 25 we have that $W - X + R = V(G)$. If $|W \cap X| \geq 2$, then we get the stronger bound $\rho_k(X \cap W) \geq 2(k-1)(k-2)$, and so in (13) our inequality improves to

$$\rho_{k,G}(W - X + R) \leq k(k-3) - 2(k-1)(k-2) + (k-2)(k+1) = 2k-6 \leq y_k,$$

a contradiction. Thus $|X \cap W| = 1$. Because R_* is not monochromatic and $|X \cap W| = 1$, there is a vertex $z \in R_* - W$. Then by (9), instead of (13) we have

$$\rho_{k,G}(W - X + R) \leq k(k-3) - (k+1)(k-2) + (k+1)(k-2) - 2k + 2 = k^2 - 5k + 2 \leq y_k,$$

a contradiction. \square

Claim 27 *Let X be a $(k-1)$ -clique, $u, v \in X$, $N(u) - X = \{a\}$, and $N(v) - X = \{b\}$. Then $a = b$.*

Proof. Assume $a \neq b$. Let $G' = G - u - v + ab$ if $ab \notin E(G)$ and $G' = G - u - v$ otherwise. By Claim 26, G' has a $(k-1)$ -coloring ϕ . Because $d(u) = d(v) = k-1$, the sets $C_a = \{1, \dots, k-1\} - \cup_{w \in N(u), w \neq v} \phi(w)$ and $C_b = \{1, \dots, k-1\} - \cup_{w \in N(v), w \neq u} \phi(w)$ each contain at least one element. Since $\phi(a) \neq \phi(b)$ and $(N(u) - a) = (N(v) - b)$, those elements must be different. Therefore ϕ can be extended to u and v . But then we have a $(k-1)$ -coloring of G , which is a contradiction. \square

Claim 28 *G does not contain $K_k - e$.*

Proof. Suppose $G[R] = K_k - e$. The only k -critical graph on k vertices is the complete graph, which is k -Ore. By assumption G is not k -Ore, so $R \neq V(G)$, but adding the missing edge to $G[R]$ creates a k -chromatic graph on R , a contradiction to Claim 26. \square

4 Clusters and sets with small potential

Definition 29 For $S \subseteq V(G)$, an S -cluster is an inclusion maximal set $R \subseteq S$ such that for every $x \in R$, $d(x) = k - 1$ and for every $x, y \in R$, $N[x] = N[y]$. A cluster is a $V(G)$ -cluster.

In this section, results on clusters will help us to derive the main lower bound on the potentials of nontrivial vertex sets, Lemma 35, which in turn will help us to prove stronger results on the structure of clusters in G .

Having the same closed neighborhood is an equivalence relation, and so the set of clusters is a partition of the set of the vertices with degree $k - 1$. Thus the following fact holds.

Fact 30 Every vertex with degree $k - 1$ is in a unique cluster.

Furthermore, if the only S -cluster is the empty set, then every vertex in S has degree at least k . By definition, if a cluster T is contained in a vertex set S , then T is also an S -cluster.

Claim 31 Every cluster T satisfies $|T| \leq k - 3$. Furthermore, for every $(k - 1)$ -clique X in G , (i) there is a unique X -cluster T (possibly $T = \emptyset$), and (ii) every non-empty X -cluster is a cluster (in other words, every cluster is either contained by X or disjoint from X). In particular, each $(k - 1)$ -clique in G contains at least 2 vertices of degree at least k .

Proof. If T is a cluster with $|T| \geq k - 2$, then $T \cup N(T) \supseteq K_k - e$, a contradiction to Claim 28.

Let X be a $(k - 1)$ -clique in G . Two distinct X -clusters would contradict Claim 27. If T is a non-empty X -cluster contained in a larger cluster T' , then each $v \in T' - X$ has to be adjacent to each vertex in X , and so G contains clique $X \cup T'$ of size at least k , a contradiction.

The final statement is proven as follows: by Fact 30 every vertex not in a cluster does not have degree $k - 1$, the minimum degree is $k - 1$, and the only cluster in X has at most $k - 3$ of the $k - 1$ vertices in X . \square

Claim 32 For every partition (A, B) of $V(G)$ with $2 \leq |A| \leq n - 2$, $|E_G(A, B)| \geq k$.

Proof. Let A_* (respectively, B_*) be the set of vertices in A (respectively, B) that have neighbors in B (respectively, A). Since G is 3-connected, $|A_*| \geq 3$ and $|B_*| \geq 3$. So by Claim 26, $G[A]$ has a $(k - 1)$ -coloring ϕ_A such that A_* is not monochromatic, and $G[B]$ has a $(k - 1)$ -coloring ϕ_B such that B_* is not monochromatic. But Gallai and Toft (see [32, p. 157]) independently proved that if $|E_G(A, B)| \leq k - 1$, then either A_* is monochromatic in every $(k - 1)$ -coloring of $G[A]$ or B_* is monochromatic in every $(k - 1)$ -coloring of $G[B]$. So, $|E_G(A, B)| \geq k$. \square

Sometimes below, our goal will be to extend to G a coloring ϕ of $G[R]$ for some R and ϕ . Recall that $Y(G, R, \phi)$ is obtained from G replacing the vertices of R with a clique whose vertices are the color classes of ϕ with at least one element in R_* . One of the ways we will control ϕ is to add edge(s) to R before we generate a $(k - 1)$ -coloring ϕ using Claim 26 and a lemma below. Our next lemma describes how edges can be placed in R so that no color class of ϕ is too large. The proof of this lemma will use the following old result of Hakimi.

Theorem 33 (Hakimi [20]) *Let (w_1, \dots, w_s) be a list of nonnegative integers with $w_1 \geq \dots \geq w_s$. Then there is a loopless multigraph F with vertex set $\{u_1, \dots, u_s\}$ such that $d_F(u_j) = w_j$ for all $j = 1, \dots, s$ if and only if $z = w_1 + \dots + w_s$ is even and $w_1 \leq w_2 + \dots + w_s$.*

For technical reasons, in one specific case of the lemma below we will allow for a hyperedge of size 3. Recall that an *independent set* in a hypergraph is a set that contains no edge: thus an independent set may contain at most 2 vertices of a hyperedge of size 3.

Lemma 34 *Let $i \geq 1$ and $s \geq 2$ be integers. Let $R_* = \{u_1, \dots, u_s\}$ be a vertex set. Then for each $z \geq 2i$ and any integral positive weight function $w : R_* \rightarrow \{1, 2, \dots\}$ such that $w(u_1) + \dots + w(u_s) = z$ and $w(u_1) \geq w(u_2) \geq \dots \geq w(u_s)$, there exists a graph H with $V(H) = R_*$ and $|E(H)| \leq i$ such that for each $1 \leq j \leq s$, $d_H(u_j) \leq w(u_j)$, and for every independent set M in H with $|M| \geq 2$,*

$$\sum_{u \in R_* - M} w(u) \geq i. \quad (14)$$

Moreover, if $s \geq 3$ and $z > 2i$, then at least one of the three stronger statements below holds:

- (i) such H with Property (14) could be chosen as a graph with at most $i - 1$ edges, or
- (ii) such H with Property (14) could be chosen as a hypergraph instead of a graph with at most $i - 1$ graph edges and one edge of size 3, or
- (iii) the weight arrangement is i -special, which means that $s = i + 1$ and $w(u_2) = \dots = w(u_s) = 1$.

Proof. The statement is trivial for $i = 1$, so assume $i \geq 2$. Consider an auxiliary integral weight function $w' : R_* \rightarrow \{1, 2, \dots\}$ such that $w'(u_1) + \dots + w'(u_s) = 2i$ and $w'(u_j) \leq w(u_j)$ for all $j = 1, \dots, s$.

Case 1: $w'(u_2) + \dots + w'(u_s) \leq i - 1$. We make $E(H) = \{u_1 u_j : 2 \leq j \leq s\}$. If M is any independent set with $|M| \geq 2$, then $u_1 \notin M$ and $w(u_1) \geq w'(u_1) \geq 2i - (i - 1)$ yielding (14). To prove the “Moreover” part in this case, observe that our H has at most $i - 1$ edges.

Case 2: $w'(u_2) + \dots + w'(u_s) \geq i$. Then by Theorem 33, there exists a loopless multigraph H' with vertex set $\{u_1, \dots, u_s\}$ such that $d_{H'}(u_j) = w'_j$ for all $j = 1, \dots, s$. We obtain a graph H from the multigraph H' by replacing each set of multiple edges with a single edge. Every independent set in H is also independent in H' . For every independent set M in H' , each of its i edges has an end outside of M , so

$$\sum_{u \in R_* - M} w(u) \geq \sum_{u \in R_* - M} w'(u) = \sum_{u \in R_* - M} d_{H'}(u) \geq |E(H')| = i.$$

This yields (14). Note that in this case, (14) holds for *every* independent set M , even if $|M| = 1$.

Now we prove the “Moreover” part of the statement. If H' had any multiple edge, then we satisfy (i) and are done. Suppose, H' is simple. Since $z > 2i$, $w'(u_\ell) < w(u_\ell)$ for some $1 \leq \ell \leq s$. If $H - u_\ell$ has an edge e , then after enlarging e to $e + u_\ell$ we still keep (14). This instance satisfies (ii), and we are done. Otherwise u_ℓ is incident to every edge of $H = H'$, and so H is a star with center u_ℓ and $i \geq 2$ edges. Each such star has only one central vertex, so every other vertex u_j satisfies $w(u_j) = w'(u_j) = d_H(u_j) = 1$. By definition, this means that the weight arrangement is i -special. So we satisfy (iii) and are done. \square

Recall that $\rho_{k, K_{k-1}}(V(K_{k-1})) = 2(k-1)(k-2)$. Importantly, this is larger than the potential of a standard set. Our main lower bound on the potentials of nontrivial vertex sets is the following.

Lemma 35 *If $R \subsetneq V(G)$ and $2 \leq |R| \leq n - 2$, then $\rho_k(R) \geq 2(k - 1)(k - 2)$. Moreover, if $\rho_k(R) = 2(k - 1)(k - 2)$, then $G[R] = K_{k-1}$.*

Proof. Assume that the lemma does not hold. Let i be the smallest integer such that there exists $R \subsetneq V(G)$ with $2 \leq |R| \leq n - 2$, $G[R] \neq K_{k-1}$ and

$$y_k + 2i(k - 1) < \rho_k(R) \leq y_k + 2(i + 1)(k - 1). \quad (15)$$

It is important that we are only minimizing i , and not necessarily minimizing $\rho_k(R)$. By Claim 26, $i \geq 1$. Since $y_k + (k + 1)(k - 1) \geq k^2 - 5k + 2 + (k + 1)(k - 1) > 2(k - 1)(k - 2)$, $i \leq \frac{k}{2}$. By the integrality, if k is odd, then $i \leq \frac{k-1}{2}$. Moreover, if $k = 4$ then $y_k = \max\{2 \cdot 4 - 6, 4^2 - 5 \cdot 4 + 2\} = 2$ and so $y_4 + 4(4 - 1) = 14 > 12 = 2(4 - 1)(4 - 2)$. Thus

$$i \leq \frac{k}{2}, \text{ moreover, if } k \text{ is odd then } i \leq \frac{k-1}{2}, \text{ and if } k = 4 \text{ then } i = 1. \quad (16)$$

Let R be a smallest set among $R \subsetneq V(G)$ with $2 \leq |R| \leq n - 2$, $\rho(R) \leq 2(k - 1)(k - 2)$ and $G[R] \neq K_{k-1}$ for which (15) holds. Since $G[R] \neq K_{k-1}$, $|R| \geq 2$, and $\rho_k(R) \leq 2(k - 1)(k - 2) = \rho_k(V(K_{k-1}))$, by Fact 11 we have $|R| \geq k$. Thus by Claim 24, for any proper $(k - 1)$ -coloring ϕ of $G[R]$, graph $Y(G, R, \phi)$ is smaller than G .

Let $Q = V(G) - R$, and for $u \in R$, let $w(u) = |N(u) \cap Q|$. By Definition 21, $R_* = \{u \in R : w(u) \geq 1\}$. Let $R_* = \{u_1, \dots, u_s\}$ and $w(u_1) \geq \dots \geq w(u_s)$. By Claim 32, $z := \sum_{i=1}^s w(u_i) = |E_G(R, V(G) - R)| \geq k$. By Claim 19, $s \geq 3$.

We will consider four cases, and the first is the main one.

Case 1: There is a $(k - 1)$ -coloring ϕ of $G[R]$ such that for every color class C of ϕ with $|C \cap R_*| \geq 2$ either

$$\sum_{u \in R_* - C} w(u) \geq i \quad (17)$$

or

$$\sum_{u \in R_* - C} w(u) = i - 1 \text{ and } \sum_{u \in C} w(u) \leq k - 2. \quad (18)$$

Let $F = Y(G, R, \phi)$ be as in Definition 21, where X is the clique replacing R .

By Claim 22, F contains a k -critical graph F' . Let $W = V(F')$ and $X' = X \cap W$. Since $|R| \geq k$, by Claim 24, $X' \neq \emptyset$ and one of the following two statements is true: (a) $F[W]$ contains a k -Ore graph, or (b) $\rho_{k,F}(W) \leq y_k$. Because $X' \neq \emptyset$, by Fact 11, $\rho_{k,F}(X') \geq (k + 1)(k - 2)$. By (9) we have

$$\rho_{k,G}(W - X + R) \leq \rho_{k,F}(W) - \rho_{k,F}(X') + \rho_{k,G}(R) \leq \rho_{k,G}(R) - 2(k - 1), \quad (19)$$

and by the choice of i , this implies that $|W - X + R| \notin [2, n - 2]$. Because $|R| \geq 2$, this means $|W - X + R| \geq n - 1$. Suppose first that $|W - X + R| = n - 1$. By Fact 20, $\rho_{k,G}(W - X + R) \geq y_k + k^2 - 3k + 4$ and so $\rho_{k,G}(R) \geq y_k + k^2 - k + 2 > 2(k - 1)(k - 2)$, contradicting the choice of R . So

$$W - X + R = V(G). \quad (20)$$

We claim that F is a k -Ore graph. We will prove this in three steps. Specifically, we will show, in order, that

$$(A) |X'| \geq 2, (B) F' \text{ is a } k\text{-Ore graph, and } (C) F' = F. \quad (21)$$

Suppose $X' = \{x_j\}$. Then $W = V(F) - X + x_j$. Let $R_j = \{u \in R_* : \phi(u) = c_j\}$. If $|R_j| = 1$, then $F \cong G[W - x_j \cup R_j]$, which is a subgraph of G . Because $|R| \geq k > 1 = |R_j|$, F is a proper

subgraph of G , but k -critical graphs do not have k -chromatic proper subgraphs. Thus $|R_j| \geq 2$. If (18) holds for R_j , then $d_{F'}(x_j) \leq \sum_{u \in R_j} w(u) \leq k-2$, but the k -critical graph F' cannot have vertices of degree less than $k-1$. Otherwise, by (17), at least i edges connect the vertices in $R_* - R_j$ with Q . Adjusting (19) to account for these edges and using (15), we have

$$\rho_{k,G}(W - \{x_j\} + R) \leq k(k-3) - (k-2)(k+1) - 2i(k-1) + \rho_{k,G}(R) = \rho_{k,G}(R) - 2(i+1)(k-1) \leq y_k,$$

which contradicts (20) and our assumption that $\rho_{k,G}(V(G)) > y_k$. This proves (A).

If F' is not a k -Ore graph, then by Claim 24, $\rho_{k,F}(W) \leq y_k$. Since every $2 \leq |X'| \leq k-1$ has potential at least $2(k-1)(k-2)$ by Fact 11, equation (9) now strengthens to

$$\rho_{k,G}(V(G)) = \rho_{k,G}(W - X + R) \leq y_k - 2(k-1)(k-2) + \rho_{k,G}(R) \leq y_k,$$

a contradiction. This proves (B). If $X' \neq X$, then the last term of (9) is nonzero and the bound in (19) reduces by $2(k-1)$. If F' is not an induced subgraph of F , then again the bound in (19) reduces by $2(k-1)$. In both cases, reducing (19) by $2(k-1)$ plus using that $|X'| \geq 2$ and the assumption $\rho_k(R) \leq 2(k-1)(k-2)$ produces

$$\rho_{k,G}(W - X + R) \leq -2(k-1) + k(k-3) - 2(k-1)(k-2) + 2(k-1)(k-2) = k^2 - 5k + 2 \leq y_k,$$

a contradiction. So, $F = F'$. This proves the claim that $F = Y(G, R, \phi)$ is k -Ore.

Suppose first that F is a k -Ore graph distinct from K_k . Let a separating set $\{x, y\}$, vertex subsets $A = A(F, x, y)$ and $B = B(F, x, y)$, and graphs $\tilde{F}(x, y)$ and $\check{F}(x, y)$ be as in Fact 13. Since $F[X']$ is a clique and $E_F(A - x - y, B - x - y) = \emptyset$, either $X' \subseteq A$ or $X' \subseteq B$. Since $xy \notin E(F)$ we may assume that either $X' \subset A - y$ or $X' \subset B - y$. Suppose first that $X' \subset A - y$. The graph $\tilde{F} - x * y$ is a subgraph of G , namely, it is $G[B - x - y]$, and by Fact 14

$$d_{\tilde{F}}(v) = d_G(v) \text{ for every } v \in B - x - y. \quad (22)$$

If $\tilde{F} - x * y$ has a vertex subset S with $|S| \geq 2$ of potential at most $(k+1)(k-2)$, then by Lemma 16, S contains a standard set S' . But each standard set S' has two vertices u and w such that $F[S'] + uw$ is not $(k-1)$ -colorable. This contradicts Claim 26. Thus $\rho_{k,\tilde{F}}(S) > (k+1)(k-2)$ for every $S \subseteq V(\tilde{F}) - x * y$ with $|S| \geq 2$. Then by Claim 18, there exists an $S \subseteq V(\tilde{F}) - x * y = B - x - y$ such that $\tilde{F}[S] \cong K_{k-1}$, and $d_{\tilde{F}}(v) = k-1$ for all $v \in S$. By (22), this contradicts Claim 31.

Now suppose that $X' \subset B - y$. Similarly to (22), the graph $\tilde{F} - x$ is a subgraph of G , namely, it is $G[A - x]$, and

$$d_{\tilde{F}}(v) = d_G(v) \text{ for every } v \in A - x - y. \quad (23)$$

As in the previous paragraph, $\rho_{k,\tilde{F}}(S) > (k+1)(k-2)$ for every $S \subseteq V(\tilde{F}) - x$ with $|S| \geq 2$. So again by Claim 18, there exists an $S' \subseteq V(\tilde{F}) - x = A - x$ such that $\tilde{F}[S'] \cong K_{k-1}$, and $d_{\tilde{F}}(v) = k-1$ for all $v \in S'$. But $|S' - y| \geq k-2$, which together with (23) contradicts Claim 31.

Thus, $F = K_k$. Let $t = |X| = |X'|$. By (21)(A), $t \geq 2$. Because $|R| \leq n-2$, $|Q| \geq 2$. So since $V(F) = X \cup Q$, we have $t \leq k-2$. Then G is obtained from $G[X]$ by adding $k-t$ vertices and at least $\binom{k}{2} - \binom{t}{2}$ edges (since a vertex in Q may be adjacent to more than one vertex in a color class of ϕ). So

$$\rho_k(V(G)) \leq \rho_k(R) + (k-t)(k+1)(k-2) - \left(\binom{k}{2} - \binom{t}{2} \right) 2(k-1). \quad (24)$$

Denote the RHS of (24) by $\mu(k, t, R)$. For fixed k and R , $\mu(k, t, R)$ is quadratic in t with a positive coefficient at t^2 , and we know that $2 \leq t \leq k - 2$. So, if $3 \leq t \leq k - 2$, then $\mu(k, t, R) \leq \max\{\mu(k, 3, R), \mu(k, k - 2, R)\}$. Furthermore,

$$\begin{aligned}\mu(k, k - 2, R) &= \rho_k(R) + 2(k + 1)(k - 2) - k(k - 1)^2 + (k - 1)(k - 2)(k - 3) \\ &\leq 2(k - 1)(k - 2) - 2k^2 + 8k - 10 = 2k - 6 \leq y_k,\end{aligned}$$

and when $3 \leq k - 2$ (i.e. $k \geq 5$),

$$\mu(k, 3, R) = \rho_k(R) + (k - 3)(k + 1)(k - 2) - k(k - 1)^2 + 6(k - 1) \leq 2(k - 1)(k - 2) - 2k^2 + 6k = 4 \leq y_k.$$

Since $\rho_k(V(G)) > y_k$, we conclude that $t = 2$ and $G[Q] = K_{k-2}$. Moreover, $\mu(4, 2, R) \leq 2(4 - 1)(4 - 2) + 20 - (6 - 1)6 = 2 = y_4$, so $k \geq 5$. Similarly to above,

$$\mu(k, 2, R) = \rho_k(R) + (k - 2)^2(k + 1) - k(k - 1)^2 + 2(k - 1) = \rho_k(R) - k^2 + k + 2. \quad (25)$$

Recall by Fact 11 that potential is always even. Thus, in order to have $\rho_k(V(G)) \geq y_k + 2$, we need

$$\rho_5(R) = 2(5 - 1)(5 - 2) = 24 \text{ and } \rho_k(R) \geq 2(k - 1)(k - 2) - 2 \text{ for } k \geq 6. \quad (26)$$

Since for $k \geq 5$, $2(k - 1)(k - 2) - 2 > y_k + 4(k - 1)$, we have $i \geq 2$. Also we conclude that each $v \in Q$ has exactly two edges in R , since otherwise the upper bound on $\rho_k(V(G))$ in (24) would be stronger by $2(k - 1)$ and together with (25) would lead to

$$\rho_k(V(G)) \leq -2(k - 1) + 2(k - 1)(k - 2) - k^2 + k + 2 = k^2 - 7k + 8 \leq y_k.$$

Let $Q = \{v_1, \dots, v_{k-2}\}$ and let $N(v_j) \cap R = \{u_{j,1}, u_{j,2}\}$ for $j = 1, \dots, k - 2$. If $\phi'(u_{j,1}) = \phi'(u_{j,2})$ for some j and some proper $(k - 1)$ -coloring ϕ' of $G[R]$, then ϕ' may be extended to all of G greedily by first coloring $Q - v_j$ and at the end coloring v_j (at each step at most $k - 2$ colors must be avoided). Similarly, if $\{\phi'(u_{j,1}), \phi'(u_{j,2})\} \neq \{\phi'(u_{j',1}), \phi'(u_{j',2})\}$ for some $j \neq j'$, then ϕ' may be extended to all of G greedily by first coloring $X - v_j - v_{j'}$ and at the end coloring v_j and $v_{j'}$. Thus for any proper $(k - 1)$ -coloring ϕ' of $G[R]$,

$$\text{for all } 1 \leq j, j' \leq k - 2, \quad \phi'(u_{j,1}) \neq \phi'(u_{j,2}) \text{ and } \{\phi'(u_{j,1}), \phi'(u_{j,2})\} = \{\phi'(u_{j',1}), \phi'(u_{j',2})\}. \quad (27)$$

Because $3 \leq s = |R_*|$, there exist distinct vertices $v', v'' \in Q$ such that $N(v') \cap R \neq N(v'') \cap R$. By symmetry, we may assume $u_{1,1} \notin N(v_2)$. Let G^* be obtained from $G[R]$ by adding edges $e_1 = u_{1,1}u_{2,1}$ and $e_2 = u_{1,1}u_{2,2}$. By (27), $\chi(G^*) \geq k$. Thus G^* contains a k -critical subgraph G° , and by the minimality of G (G° has fewer vertices), G° is k -Ore or $\rho_{k,G^*}(V(G^\circ)) \leq y_k$. Since $i \geq 2$ and we have added at most two edges (e_1 or e_2 may belong to G), by (15) and the minimality of i , $\rho_{k,G^*}(V(G^\circ)) \geq \rho_{k,G}(V(G^\circ)) - 4(k - 1) > y_k$, and so G° is k -Ore. Moreover, in this case $\rho_{k,G^*}(V(G^\circ)) = k(k - 3)$, and so $\rho_{k,G}(V(G^\circ)) \leq k(k - 3) + 4(k - 1) \leq y_k + 3(2(k - 1))$. Hence $V(G^\circ)$ satisfies (15) for some $i \leq 2$. By the minimality of i and of $|R|$, this gives

$$i = 2, V(G^\circ) = R \text{ and } G[R] = G^\circ - e_1 - e_2. \quad (28)$$

Also, since $i \geq 2$ and $(k + 1)(k - 2) \leq y_k + 2(2(k - 1))$,

$$G[R] \text{ contains no set with potential at most } (k + 1)(k - 2). \quad (29)$$

For all $S \subseteq V(G^\circ) - u_{1,1}$, we have $G^*[S] \cong G[S]$. Thus, by (28) and (29), Claim 18 applies to $G^\circ = G^*$ and $u_{1,1}$. By this claim, $G^\circ - u_{1,1}$ contains a clique S of order $k - 1$ such that each vertex in S has degree $k - 1$. Since $u_{1,1} \notin S$, S is also clique in G . Since $e_1, e_2 \subset N(Q)$, if $u \in S - N(Q)$ then $d_G(u) = k - 1$. Because $N(Q)$ is 2-colorable, this implies that there is an $S' \subset S$ with $|S'| \geq k - 3$ such that $d_G(u) = k - 1$ for all $u \in S'$. Each vertex of S' is in a cluster by Fact 31, and Claim 31 says that all of S' is one cluster and that $|S'| = k - 3$. Let $\{u'\} = N(S') - S$. Then $\rho_{k,G}(S \cup u') \leq k^2 + k - 4 < y_k + 3(2(k - 1))$. By the minimality of R , we have $R = S + u'$. So $u_{1,1} = u'$ and $G[R]$ is a k -clique minus the edges $e_1 = u_{1,1}u_{2,1}$ and $e_2 = u_{1,1}u_{2,2}$. But then for any possible choice of $u_{1,2}$, there exists a $(k - 1)$ -coloring ϕ of $G[R]$ such that $\{\phi(u_{1,1}), \phi(u_{1,2})\} \neq \{\phi(u_{2,1}), \phi(u_{2,2})\}$. This contradiction to (27) finishes Case 1.

In all subsequent cases, we will use Lemma 34 in order to construct either a $(k - 1)$ -coloring of G or a $(k - 1)$ -coloring of $G[R]$ fitting into Case 1. For the rest of the proof, we denote $z = \sum_{u \in R_*} w(u) = |E(R, Q)| \geq k$ and assume that Case 1 does not hold.

Case 2: $2i \geq z = |E(R, Q)|$. By (16), in order to have $2i \geq |E(R, Q)|$, we need $i = \frac{k}{2}$, $k \geq 6$, and $|E(R, Q)| = k$. For $k \geq 6$, we know that $y_k = k^2 - 5k + 2$. By Lemma 34 for $i - 1$ instead of i , we can add to $G[R_*]$ a set E_1 of at most $i - 1$ edges such that (14) holds with $i - 1$ instead of i . By (15), $\rho_{k,H_1}(R') > y_k + 2k - 2 = k(k - 3)$ for every $R' \subseteq R$ with $|R'| \geq 2$. So, by Corollary 9, H_1 has a $(k - 1)$ -coloring ϕ . Since Case 1 does not hold, ϕ has a color class C that satisfies neither (17) nor (18). This means that $\sum_{u \in R_* - C} w(u) = i - 1$ and $\sum_{u \in C} w(u) \geq k - 1$. But then $|E(R, Q)| \geq k - 1 + i - 1 \geq k - 1 + \frac{k}{2} - 1 = \frac{3k}{2} - 2$. Since $k \geq 6$, this contradicts $|E(R, Q)| = k$.

If Case 2 does not hold, then $z > 2i$ and, since $s = |R_*| \geq 3$, the “moreover” part of Lemma 34 holds.

Case 3: The set $\{w(u_1), \dots, w(u_s)\}$ is i -special: $s = i + 1$ and $w(u_2) = \dots = w(u_s) = 1$. This means that many (exactly $z - i \geq i$) edges connect u_1 with Q and each of the vertices u_2, \dots, u_{i+1} is connected to Q by exactly one edge. For $j = 2, \dots, i + 1$, let q_j be the vertex in Q such that $u_j q_j \in E(G)$. Let $E_0 = \{u_1 u_j : 2 \leq j \leq i\}$ and $H_0 = G[R] \cup E_0$. Since $|R| < n$, H_0 is smaller than G . Since $|E_0| = i - 1$, by (15), $\rho_{k,H_0}(R') > y_k + 2k - 2 \geq k(k - 3)$ for every $R' \subseteq R$ with $|R'| \geq 2$. So, by the second part of Corollary 9, H_0 has a proper $(k - 1)$ -coloring ϕ . By construction, ϕ is a proper $(k - 1)$ -coloring of $G[R]$ that satisfies $\phi(u_j) \neq \phi(u_1)$ for each $2 \leq j \leq i$. If $\phi(u_{i+1}) \neq \phi(u_1)$, then for every monochromatic subset M of R_* in $G \cup E_0$ with $|M| \geq 2$, (14) holds. This contradicts (17), so suppose $\phi(u_{i+1}) = \phi(u_1)$.

Let G_0 be obtained from $G[V(G) - (R - u_1)]$ by adding edge $u_1 q_{i+1}$. By Claim 26, G_0 has a $(k - 1)$ -coloring ϕ' . Since $i \leq \frac{k}{2}$, we can rename the colors in ϕ' so that $\phi'(u_1) = \phi(u_1) = \phi(u_{i+1})$ and $\phi(\{u_2, \dots, u_i\}) \cap \phi'(\{q_2, \dots, q_i\}) = \emptyset$. Then $\phi \cup \phi'$ is a proper $(k - 1)$ -coloring of G , a contradiction.

Case 4: The set of weights $\{w(u_1), \dots, w(u_s)\}$ is not i -special and $2i < z$, so that Part (i) or (ii) of the “moreover” part of Lemma 34 holds. If Part (i) holds, then we take this set E_0 of at most $i - 1$ edges and let $H_0 = G[R] + E_0$. In this case by (15), $\rho_{k,H_0}(R') > y_k + 2k - 2 \geq k(k - 3)$ for every $R' \subseteq R$ with $|R'| \geq 2$. So, by the second part of Corollary 9, H_0 has a $(k - 1)$ -coloring ϕ , satisfying (17) of Case 1.

Suppose now that Part (ii) holds: *there is a hypergraph H with at most $i - 1$ graph edges and a 3-edge $e_0 = \{u, v, w\}$ such that $d_H(u_j) \leq w(u_j)$ for all $j = 1, \dots, s$ and (14) holds.* Let H_1 be obtained from $G[R]$ by adding the set of edges $E(H) - e_0$ and edge uv . Since $|R| < n$, H_1 is smaller than G . A proper $(k - 1)$ -coloring of H_1 would satisfy (17) of Case 1, so $\chi(H_1) \geq k$. Then H_1 has a k -critical subgraph H'_1 . Let $R' = V(H'_1)$. If H'_1 is not a k -Ore graph, then by the minimality of

G , $\rho_{k,H_1}(R) \leq y_k$ and so $\rho_{k,G}(R') \leq y_k + 2i(k-1)$, contradicting the minimality of i . Thus, H'_1 is a k -Ore graph and $\rho_{k,H_1}(R') = k(k-3) \leq y_k + 2k - 2$. Then $\rho_{k,G}(R') \leq \rho_{k,H_1}(R') + 2i(k-1) \leq y_k + 2(i+1)(k-1)$, and by the minimality of R , $R' = R$. Furthermore, if $H'_1 \neq H_1$, then it has the same vertex set as H_1 and at least one fewer edge, in which case,

$$\rho_{k,G}(R) \leq \rho_{k,H'_1}(R) + 2i(k-1) \leq \rho_{k,H_1}(R) + 2(i-1)(k-1) \leq k(k-3) + 2(i-1)(k-1) \leq y_k + 2i(k-1),$$

a contradiction to (15). So, H_1 is a k -Ore graph and so $\rho_{k,G}(R) = k(k-3) + 2i(k-1)$. By the minimality of i and R , any $W \subset R$ such that $|W| \geq 2$ satisfies $\rho_{k,G}(W) > \rho_{k,G}(R)$. Graph $H_1 - uv$ is $G[R]$ plus $i-1$ edges, so for any $W \subset V(H'_1)$ with $|W| \geq 2$ we have

$$\rho_{k,H_1-uv}(W) \geq \rho_{k,G}(W) - 2(i-1)(k-1) = k(k-3) + 2(k-1) = (k+1)(k-2).$$

Thus by Lemma 17, $H_1 - uv$ has a $(k-1)$ -coloring ϕ with $\phi(w) \neq \phi(u)$. This is a $(k-1)$ -coloring of H_0 , satisfying (17) of Case 1. \square

Recall that a standard set has potential $(k+1)(k-2)$. Because $(k+1)(k-2) < 2(k-1)(k-2)$ when $k \geq 4$, Lemma 35 implies that G cannot contain a standard set of size at most $n-2$. So if we find a standard set after some modifications made to G , then we know that this set contains vertices affected by the modifications. This claim will be a useful tool when used in conjunction with Claim 18 (i.e. when $E' = \emptyset$ and $|S'| = 1$).

Corollary 36 *Let H be a subgraph of G . Let H' be a graph that contains H as a subgraph (but possibly itself is not a subgraph of G), that is $H' = H + S' + E'$, where S' is a set of vertices and E' is a set of edges that have been added. If $S \subseteq V(H)$ with $2 < |S| \leq n-2$ and each $e \in E'$ satisfies $e \not\subseteq S$, then we have $\rho_{k,H'}(S) > (k+1)(k-2)$. In other words, if H' contains a set S with $2 < |S| < n-2$ and $\rho_{k,H'}(S) \leq (k+1)(k-2)$, then $S \cap S' \neq \emptyset$ or there is an $e \in E'$ such that $e \subset S$.*

Claim 37 *If v is not in a $(k-1)$ -clique X , then $|N(v) \cap X| \leq \frac{k-1}{2}$. Furthermore, if T is a cluster in a $(k-1)$ -clique X , then $|T| \leq \frac{k-1}{2}$.*

Proof. If $|N(v) \cap X| \geq \lceil k/2 \rceil$, then $\rho_k(X+v) \leq 2(k-2)(k-1) - 2$. Since $n \geq k+2$, this contradicts Lemma 35. This proves the first part.

Suppose now that T is a cluster in a $(k-1)$ -clique X . Since $|X| = k-1$ and $d(w) = k-1$ for every $w \in T$, each such w has the unique neighbor $v(w)$ outside of X . But by the definition of a cluster, $v(w)$ is the same, say v , for all $w \in T$. This means that $T \subseteq X \cap N(v)$, so $|N(v) \cap X| \geq |T|$. Thus the second part follows from the first. \square

Claim 38 *Suppose T is a cluster in G , $t = |T| \geq 2$, and $N(T) \cup T$ contains a $(k-1)$ -clique X . Then $d_G(v) \geq k-1+t$ for every $v \in X-T$.*

Proof. Suppose $v \in X-T$ and $d(v) \leq k-2+t$. Recall that every vertex of degree $k-1$ is in a cluster, by Claim 31(ii) every cluster that intersects X is contained by X , and by Claim 31(i), X contains only one nonempty cluster, namely, T . So v is not in a cluster and thus by Fact 30, $d(v) \geq k$.

By the definition of a cluster, each vertex in T has degree $k - 1$ and has identical closed neighborhoods, so $|T \cup N(T)| = k$. By this and Claim 28, T is contained in at most one $(k - 1)$ -clique (which is X), and so

$$N(T) \cup T - v \text{ does not contain } K_{k-1}. \quad (30)$$

Because T and v are parts of the same clique, $|N(v) - T| = d(v) - |T|$, and by assumption this is at most $k - 2$. Let $u \in T$ and $G' = G - v + u'$, where u' is a new vertex that satisfies $N[u'] = N[u]$. Suppose G' has a $(k - 1)$ -coloring $\phi' : V(G') \rightarrow C = \{c_1, \dots, c_{k-1}\}$. Then there is a $(k - 1)$ -coloring ϕ of G as follows: set $\phi|_{V(G)-T-v} = \phi'|_{V(G')-T-u'}$, $\phi(v) \in C - \phi'(N(v) - T)$, and then color T using colors in $\phi'(T \cup u') - \phi(v)$. This is a contradiction, so there is no $(k - 1)$ -coloring of G' . Thus G' contains a k -critical subgraph G'' . Let $W = V(G'')$. By Corollary 9, $\rho_{k,G'}(W) \leq k(k - 3)$.

By the criticality of G , graph G'' is not a subgraph of G . So $u' \in W$. By symmetry, we have $T \subset W$. But then

$$\rho_{k,G}(W - u') \leq k(k - 3) - (k - 2)(k + 1) + 2(k - 1)(k - 1) = 2(k - 2)(k - 1).$$

This implies by Lemma 35 that either $G[W - u']$ is a K_{k-1} or $W - u' = V(G) - v$. If the former holds, then because $G[W - u']$ is a complete graph and $T \subset W - u'$ we have $N(T) \cup T \supset G[W - u'] \cong K_{k-1}$, and because $v \notin W$ this is a contradiction to (30). If the latter holds, then we have a contradiction to Fact 20, since $d(v) \geq k$. \square

Claim 39 *Let $xy \in E(G)$, $N[x] \neq N[y]$, x is in a cluster of size s , y is in a cluster of size t , and $s \geq t$. Then x is in a $(k - 1)$ -clique. Furthermore, $t = 1$.*

Proof. Assume that x is not in a $(k - 1)$ -clique. Let $G' = G - y + x'$ for new vertex x' , where $N[x'] = N[x]$. By the definition of a cluster, $d(x) = d(y) = k - 1$. Both G' and G have the same number of vertices and the same number of edges (because $xy \in E(G)$, vertex x lost a neighbor in y and gained a neighbor in x'), so by Rule (S3), G' is smaller than G . If G' has a $(k - 1)$ -coloring $\phi' : V(G') \rightarrow C = \{c_1, c_2, \dots, c_{k-1}\}$, then we extend it to a proper $(k - 1)$ -coloring ϕ of G as follows: define $\phi|_{V(G)-x-y} = \phi'|_{V(G')-x-x'}$, then choose $\phi(y) \in C - (\phi'(N(y) - x))$, and $\phi(x) \in \{\phi'(x), \phi'(x')\} - \{\phi(y)\}$.

So, $\chi(G') \geq k$ and G' contains a k -critical subgraph G'' . Let $W = V(G'')$. By criticality of G and because $y \notin G''$, we have that $G'' \neq G$ and G'' is not a subgraph of G . Since G'' is not a subgraph of G , $x' \in W$. By symmetry, $x \in W$. Because $d(x') = k - 1$, we have

$$\rho_{k,G}(W - x') \leq k(k - 3) - \rho_{k,G'}(\{x'\}) + 2(k - 1)d(x') = 2(k - 2)(k - 1). \quad (31)$$

By assumption, x is not in a $(k - 1)$ -clique, so Lemma 35 implies that $|W - x'| > n - 2$. Thus $W - x' = V(G) - y$, which implies $V(G') = V(G'')$ and that $|W - x'| = n - 1$. By Corollary 9, $\rho_{k,G''}(W) \leq k(k - 3)$. Moreover, because G'' is smaller than G' which is smaller than G , we have by the minimality of G that if G'' is not k -Ore then $\rho_{k,G''}(W) \leq y_k$. If $G'' \neq G'$ then $\rho_{k,G''}(W) - 2(k - 1) \geq \rho_{k,G'}(W)$. Both of these statements, when used to strengthen (31), contradicts Fact 20. So $G'' = G'$ and G'' is k -Ore, which combined implies that G' is a k -Ore graph.

Since $n > k$, $G' \neq K_k$. Let the separating set $\{u, v\}$, vertex subsets $A = A(G', u, v)$ and $B = B(G', u, v)$, and graphs $\tilde{G}'(u, v)$ and $\tilde{G}''(u, v)$ be as in Fact 13. By Corollary 36, because A is

a standard set, we have $x' \in A$. Therefore $x' \notin V(\tilde{G}'(u, v)) - u * v$. We now apply Corollary 36 to $\tilde{G}'(u, v)$ to see that $\rho_{k, \tilde{G}'(u, v)}(W) > (k+1)(k-2)$ for every $W \subseteq V(\tilde{G}') - u * v$ with $|W| \geq 2$. Then by Claim 18, there exists a $S \subseteq V(\tilde{G}'(u, v)) - u * v$ such that $\tilde{G}'(u, v)[S] \cong K_{k-1}$, and $d_{\tilde{G}'(u, v)}(w) = k-1$ for all $w \in S$. By Claim 31, vertex y in G is adjacent to at most $k-3$ vertices in S . By Fact 14.5, the vertices in $S - N(y)$ have degree $k-1$ in G , so S contains a cluster T , and $|T| \geq 2$. Then by Claim 38, the degree of each vertex in $S - T$ in G is at least $k+1$. This is impossible, since each of them has in G at most one extra neighbor (and it is y) in comparison with $\tilde{G}'(u, v)$. This proves the first part: x is in a $(k-1)$ -clique, say X .

Let T_y be the cluster containing y . By the definition of a cluster, every vertex in T_y has the same neighbors as y , and so $T_y \subseteq N(x)$. Clearly, the clique X containing x is a part of $N[x]$. The second part follows from the fact that by Claim 31, $T_y \cap X = \emptyset$, and so $|T_y| \leq |N(x) - X| = d(x) - (k-2) = 1$. \square

Claim 40 *Suppose T is a cluster in G , $t = |T| \geq 2$, and $N(T) \cup T$ does not contain K_{k-1} . Then $d_G(v) \geq k-1+t$ for every $v \in N(T) - T$.*

Proof. By Claim 39, $k \leq d(v)$. Now the proof follows exactly as the proof to Claim 38. \square

5 Proof of Theorem 10

Now we are ready to prove the theorem. Recall that G is a minimal according to relation "smaller" counterexample to our theorem: it is a k -critical graph with $\rho_k(V(G)) > y_k$ and is not k -Ore.

We will use the following result on k -critical graphs which is Corollary 9 in [26].

Lemma 41 ([26]) *Let G be a k -critical graph. Let disjoint vertex subsets A and B be such that*

- (a) *either A or B is independent;*
- (b) *$d(a) = k-1$ for every $a \in A$;*
- (c) *$d(b) = k$ for every $b \in B$;*
- (d) *$|A| + |B| \geq 3$.*

Then (i) $e(G(A, B)) \leq 2(|A| + |B|) - 4$ and (ii) $e(G(A, B)) \leq |A| + 3|B| - 3$.

5.1 Case $k = 4$

In this subsection we prove the theorem for $k = 4$. Specifically, we will prove that $|E(G)| \geq \frac{5}{3}|V(G)|$, which will imply that $\rho_{4, G}(V(G)) \leq y_4 = 2$.

Claim 42 *Each vertex with degree 3 has at most 1 neighbor with degree 3.*

Proof. Let x be such that $N(x) = \{a, b, c\}$ and $d(a) = 3$. Then x and a are each in a cluster. Because no cluster is larger than $k-3 = 1$ by Claim 31, a and x are in different clusters. Then by Claim 39, $G[\{x, b, c\}]$ is a K_3 . So by Claim 31, $d(b), d(c) \geq 4$. \square

We now use discharging to show that $|E(G)| \geq \frac{5}{3}n$. Each vertex begins with charge equal to its degree. If $d(v) \geq 4$, then v gives charge $\frac{1}{6}$ to each neighbor. Note that v will be left with charge at

least $\frac{5}{6}d(v) \geq \frac{10}{3}$. By Claim 42, each vertex of degree 3 will end with charge at least $3 + \frac{2}{6} = \frac{10}{3}$. Therefore the total charge is at least $\frac{10}{3}n$, and thus so is the sum of the vertex degrees. Hence the number of edges is at least $\frac{5}{3}n$. \square

5.2 Case $k = 5$

In this subsection we prove the theorem for $k = 5$. Specifically, we will prove that $|E(G)| \geq \frac{9}{4}|V(G)|$, which will imply that $\rho_{5,G}(V(G)) \leq 0 < y_5 = 4$.

Claim 43 *If $k = 5$, then each cluster has only one vertex.*

Proof. Suppose the claim does not hold. By Claim 31, every cluster has size at most $k - 3 = 2$, so assume that $\{x, y\}$ is a cluster: $N[x] = N[y]$ and $d(x) = d(y) = 4$. Let $N(x) = \{y, a, b, c\}$. By assumption G is not 5-Ore and therefore G is not K_5 (and since it is critical, it does not contain a k_5). By Claim 26, G does not contain a subgraph isomorphic to $K_5 - e$. Therefore any five vertices in G induce at most $\binom{5}{2} - 2$ edges, and thus $|E(G[\{a, b, c\}])| \leq 1$. By Claims 38 and 40, we can rename the vertices in $\{a, b, c\}$ so that $ab, ac \notin E(G)$ and $d(c) \geq 6$.

We obtain G' from G by deleting x and y and gluing a with b . If G' is 4-colorable, then so is G . This is because a 4-coloring of G' will have at most 2 colors on $N[x] - \{x, y\}$ and therefore could be extended greedily to x and y .

So G' contains a k -critical subgraph G'' . Let $W' = V(G'')$. Then by Corollary 9, $\rho_{5,G'}(W') \leq 10$. Furthermore, because G'' is smaller than G , if G'' is not k -Ore, then $\rho_{5,G'}(W') \leq 4$.

Because G is critical and $x, y \notin G'' \subseteq G'$, graph G'' is not a subgraph of G . This implies that $a * b \in G''$. Let $W = W' - a * b + a + b + x + y$. If c is not in W' , then by construction W has 3 more vertices and induces at least 5 more edges than W' . If c is in W' , then W has 3 more vertices and at least 7 more edges compared to W' . Suppose first that $c \notin W'$, so that $\rho_{5,G}(W) \leq 10 + 54 - 40 = 24$. Because $ab \notin E(G)$, $G[W]$ is not a K_4 . By Lemma 35, $|W| \geq n - 1$. Therefore $W = V(G) - c$ and $\rho_{5,G}(W) \leq 24$, but this contradicts Fact 20 because $d(c) \geq 6 = k + 1$.

So now we assume that $c \in W'$, which means that $\rho_{5,G}(W) \leq \rho_{5,G'}(W') + 54 - 56 \leq 8$. By Lemma 25, $W = V(G)$, which then implies $V(G'') = V(G')$. Furthermore, if G'' is not k -Ore, then as mentioned above the bound on $\rho_{5,G'}(W')$ changes from 10 to 4 giving us an extra -6 . If G'' is a proper subgraph of G' , then we missed an edge in our calculation of $\rho_{5,G}(W)$ and we have an extra -8 . In either case we save at least an extra -6 , and our bound become $\rho_{k,V(G)} = \rho_{5,G}(W) \leq 8 - 6 = 2 < y_k$, a contradiction to the choice of G . So G' is k -Ore. This also implies that $N(a) \cap N(b) = \{x, y\}$, because $G'' = G'$ and critical graphs do not have multi-edges, so we would have gained an extra edge when we undo the merge of a and b into $a * b$, which could have saved an extra -8 yielding to the same contradiction.

Since $d(c) \geq 6$, G' cannot be K_5 . Let the separating set $\{u, v\}$, vertex subsets $A = A(G', u, v)$ and $B = B(G', u, v)$, and graphs $\tilde{G}'(u, v)$ and $\check{G}'(u, v)$ be as in Fact 13. By Fact 14 $\rho_{k,G'}(A) = (k + 1)(k - 2)$, by Corollary 36, $a * b \in A$. Therefore $G'[B - x - y] \subset G$ and so $V(\check{G}') - V(G) = u * v$. We now apply Corollary 36 to $\check{G}'(u, v)$ to see that $\rho_{k,\check{G}'(u,v)}(W) > (k + 1)(k - 2)$ for every $W \subseteq V(\check{G}') - u * v$ with $|W| \geq 2$. Then by Claim 18, there exists an $S \subseteq V(\check{G}'(u, v)) - u * v$ such that $\check{G}'(u, v)[S] \cong K_{k-1}$, and $d_{\check{G}'(u,v)}(w) = k - 1 = 4$ for all $w \in S$.

We propose that each vertex in $S - c$ has degree $k - 1$ in G . Note that this would imply that every vertex in $S - c$ is in a cluster by Fact 30. Because S is a $(k - 1)$ -clique, by Claim 31 there

is only one cluster in S , so the proposition implies that $T = S - c$ is a cluster and $|T| \geq 3$, which contradicts Claim 31 that each cluster in G has size at most $k - 3 = 2$. This will complete the proof.

So now we prove the claim, and in order to do that we must understand how it is possible that vertices in S could have larger degree in G than in \check{G}' . By Fact 14(7), they do not grow in degree from \check{G}' to G' . Because $N(a) \cap N(b) = \{x, y\}$, the only vertices that grow in degree from G' to G are a, b, c . But we already showed that $a * b \notin V(\check{G}') - u * v$ and $S \subseteq V(\check{G}'(u, v)) - u * v$, so it must be that $S \cap \{a, b, c\} \subseteq \{c\}$. \square

Claim 44 *Each K_4 -subgraph of G contains at most one vertex with degree 4. Furthermore, if $d(x) = d(y) = 4$ and $xy \in E(G)$, then each of x and y is in a K_4 .*

Proof. Each vertex of degree 4 is in a cluster by definition, and by Claim 31, each K_4 contains only one cluster. The first statement of our claim then follows from Claim 43 and the second — from Claim 39. \square

Definition 45 *Let $H \subseteq V(G)$ be the set of vertices of degree 5 not in a K_4 , and $L \subseteq V(G)$ be the set of vertices of degree 4 not in a K_4 . Set $\ell = |L|$, $h = |H|$ and $e_0 = |E(L, H)|$.*

Claim 46 $e_0 \leq 3h + \ell$.

Proof. This is trivial if $h + \ell \leq 2$. By Claim 44, L is independent. So the claim follows by Lemma 41(ii) with $A = L$ and $B = H$. \square

We will now use discharging to show that $|E(G)| \geq \frac{9}{4}n$, which will finish the proof to the case $k = 5$. Let every vertex $v \in V(G)$ have initial charge $d(v)$. The discharging has one rule:

Rule R1: Each vertex in $V(G) - H$ with degree at least 5 gives charge $1/6$ to each neighbor.

We will show that the charge of each vertex in $V(G) - H - L$ is at least 4.5, and then show that the average charge of the vertices in $H \cup L$ is at least 4.5.

Claim 47 *After discharging, each vertex in $V(G) - H - L$ has charge at least 4.5.*

Proof. Let $v \in V(G) - H - L$. If $d(v) = 4$ and $v \notin L$, then v is in a K_4 and by Claim 44 v receives charge $1/6$ from at least 3 neighbors and gives no charge. If $d(v) = 5$ and $v \notin H$, then v is in a K_4 and by Claim 44 $N(v)$ contains at least 2 vertices with degree at least 5. Therefore v gives charge $1/6$ to 5 neighbors, but receives charge $1/6$ from at least 2 neighbors. If $d(v) \geq 6$, then v is left with charge at least $5d(v)/6 \geq 4.5$. \square

Claim 48 *After discharging, the sum of the charges on the vertices in $H \cup L$ is at least $4.5|H \cup L|$.*

Proof. By Claim 44, if $v \in L$ then every vertex in $N(v)$ has degree at least 5. By Rule R1, vertices in L receive from outside of $H \cup L$ the charge at least $\frac{1}{6}(4\ell - |E(H, L)|)$. By Claim 46, $|E(H, L)| \leq 3h + \ell$. So, the total charge on $H \cup L$ is at least

$$5h + 4\ell + \frac{1}{6}(4\ell - (3h + \ell)) = 4.5(h + \ell),$$

as claimed. \square

Combining Claims 47 and 48, the total charge is at least $\frac{9}{2}n$. Thus the sum of vertex degrees is at least $\frac{9}{2}n$, and so $|E(G)| \geq \frac{9}{4}|V(G)|$. \square

5.3 Case $k \geq 6$

In this subsection we prove Theorem 10 for $k \geq 6$. We will prove that $|E(G)| \geq \frac{(k+1)(k-2)}{2(k-1)}|V(G)|$, which will imply that $\rho_{k,G}(V(G)) \leq 0 \leq y_k = k^2 - 5k + 2$. This proof will involve several claims.

Claim 49 *Suppose $k \geq 6$, X is a $(k-1)$ -clique, and $v \in X$ has degree $k-1$. Then X contains at least $(k-1)/2$ vertices with degree at least $k+1$.*

Proof. Let $\{u\} = N(v) - X$. Assume that X contains at least $k/2$ vertices with degree at most k . By Claim 37 $|N(u) \cap X| < k/2$, so there exists a $w \in X$ such that $uw \notin E(G)$ and $d(w) \leq k$. By Claim 27, $d(w) = k$, so assume $N(w) - X = \{a, b\}$. Let G' be obtained from $G - v$ by adding edges ua and ub .

Suppose G' has a $(k-1)$ -coloring f . If $f(u)$ is not used on $X - w - v$, then we recolor w with $f(u)$. So, v will have at least two neighbors of color $f(u)$, and we can extend the $(k-1)$ -coloring to v .

Thus G' is not $(k-1)$ -colorable and so contains a k -critical subgraph G'' . Let $W = V(G'')$. By Corollary 9, $\rho_{k,G'}(W) \leq k(k-3)$ and so $\rho_{k,G}(W) \leq k(k-3) + 2(k-1)(2) = k^2 + k - 4 < 2(k-2)(k-1)$. If $W \neq V(G')$ then this contradicts Lemma 35, since in this case $|W| \leq |V(G')| - 1 \leq n - 2$. So, $W = V(G')$.

If G'' is not a k -Ore graph, then by the minimality of G , $\rho_{k,G''}(W) \leq y_k$, and since edges only reduce potential we have $\rho_{k,G'}(W) \leq \rho_{k,G''}(W)$, and so

$$\rho_{k,G}(V(G)) \leq \rho_{k,G'}(W) + (k-2)(k+1)(1) - 2(k-1)(k-3) < y_k$$

when $k \geq 6$. If $G'' \neq G'$, then we did not account for an edge and thus $\rho_{k,G'}(W) \leq \rho_{k,G''}(W) - 2(k-1)$, which leads to the same contradiction because by Fact 11 $\rho_{k,G''}(W) - 2(k-1) = k(k-3) - 2(k-1) \leq y_k$. So, our case is that G'' is a k -Ore graph, $G'' = G'$, and so G' is a k -Ore subgraph. Since $G' - ua - ub$ is a subgraph of G , by Corollary 36 $\rho_{k,G'}(U) > (k+1)(k-2)$ for every $U \subseteq V(G') - u$ with $|U| \geq 2$. Then by Claim 18, there exists a $S \subseteq V(G') - u$ such that $G'[S] \cong K_{k-1}$, and $d_{G'(v)} = k-1$ for all $v \in S$. But for every $z \in S - a - b$, $d_G(z) = d_{G'}(z)$. This implies that there is a cluster of size at least $k-1-2$ in S which is a $(k-1)$ -clique, which contradicts Claim 37 because $k-3 > \frac{k-1}{2}$ when $k \geq 6$. \square

Claim 50 *If $k = 6$ and a cluster T is contained in a 5-clique X , then $|T| = 1$.*

Proof. By Claim 37, assume that $T = \{v_1, v_2\}$. Let $N(v_1) - X = \{y\}$ and $\{u, u', u''\} = X - T$. By Claim 49, $d(u), d(u'), d(u'') \geq 7$. Obtain G' from $G - T$ by gluing u to y .

Suppose that G' has a 5-coloring. Then we can extend this coloring to a coloring of G by greedily assigning colors to T , because only 3 different colors appear on the set $\{u, u', u'', y\}$. So we may assume that $\chi(G') \geq 6$. Then G' contains a 6-critical subgraph G'' . Let $W = V(G'')$. Then by

Corollary 9, $\rho_{6,G'}(W) \leq 6(6-3) = 18$. Since G'' is not a subgraph of G because G itself is critical, $u * y \in W$. Let $t = |\{u', u''\} \cap W|$.

Case 1: $t = 0$. Then $\rho_{6,G}(W - u * y + y + X) \leq 18 + 28(5) - 10(12) = 38$. By Lemma 35, $|W - u * y + y + X| \geq n - 1$. We did not account for edges in $E(\{u', u''\}, V(G) - X)$, and each of u', u'' has at least 3 neighbors outside of X . Thus $\rho_{6,G}(W - u * y + y + X) \leq 38 - 10 \cdot 4 < 0$.

Case 2: $t = 1$. Then $\rho_{6,G}(W - u * y + y + u + T) \leq 18 + 28(3) - 10(7) = 32$. By Lemma 35, $|W - u * y + y + u + T| \geq n - 1$, so $W - u * y + y + u + T$ is either $V(G) - u'$ or $V(G) - u''$. But because $d(u') \geq 7 = k + 1$ (symmetrically for u''), Fact 20 says that $\rho_{6,G}(V(G) - u') > y_k + k^2 + k = 50$, which is a contradiction.

Case 3: $t = 2$. Then $\rho_{6,G}(W - u * y + y + u + T) \leq 18 + 28(3) - 10(9) = 12$. By Lemma 35 such a set can not have size less than $n - 1$ (the other option, that it has size at most 2, is invalid because we added four things to it) and by Fact 20 such a set can not have size $n - 1$ as those sets - regardless of the degree of the single vertex missing - have potential more than $y_k + k^2 - 3k + 4 = 26$. So it has size n and therefore $W - u * y + y + u + T = V(G)$. If G'' is not k -Ore, then by minimality of G , $\rho_{k,G''}(W) \leq y_k$ and we save an extra -10 in the calculation of the potential. If $G'' \neq G'$, then we did not account for an edge, and we save an extra -10 in the calculation of the potential. In either case, instead of 12 the above calculation becomes $\rho_{6,G}(W - u * y + y + u + T) \leq 2 < y_6$, which contradicts our choice of G . So $G' = G''$ and is 6-Ore.

Since $G'' - u * y$ is a subgraph of G , by Corollary 36 $\rho_{k,G'}(U) > (k+1)(k-2)$ for every $U \subseteq V(G') - u * y$ with $|U| \geq 2$. Then by Claim 18, there exists a $S \subseteq V(G') - u * y$ such that $G'[S] \cong K_5$, and $d_{G'(v)} = 5$ for all $v \in S$. By Fact 30 each vertex with degree $k - 1$ in G is in a cluster, and by Claim 37, at most 2 vertices in S are in clusters. So in S there exists at least three vertices $z_1, z_2, z_3 \in S$ such that $d_G(z_i) \neq d_{G'}(z_i)$ for $1 \leq i \leq 3$. But the only vertices whose degree shrinks from G to G' are of two types: (a) those in $N[v_1] = N[v_2] = \{y, v_1, v_2, u, u', u''\}$ and (b) those in $N(y) \cap N(u)$. Because $T = \{v_1, v_2\}$ was deleted, and $u * y \notin S$, we have at most two vertices of type (a). Then we must have had a vertex of type (b), but then we get an extra edge from G' to G that was not counted before when we calculated the potential. This extra edge causes a contradiction for the same reason as the contradiction from when $G' \neq G''$ gave us an extra edge. \square

Definition 51 We partition $V(G)$ into four classes: L_0 , L_1 , H_0 , and H_1 . Let H_0 be the set of vertices with degree k , H_1 be the set of vertices with degree at least $k + 1$, and $H = H_0 \cup H_1$. Let

$$L = \{u \in V(G) : d(u) = k - 1\},$$

$$L_0 = \{u \in L : N(u) \subseteq H\},$$

and

$$L_1 = L - L_0.$$

Set $\ell = |L_0|$, $h = |H_0|$ and $e_0 = |E(L_0, H_0)|$.

Claim 52 $e_0 \leq 2(\ell + h)$.

Proof. This is trivial if $h + \ell \leq 2$. By definition, L_0 is independent. The claim follows by applying Lemma 41(i) for $A = L$ and $B = H$ for $h + \ell \geq 3$. \square

Let every vertex $v \in V(G)$ have initial charge $d(v)$. Our discharging has two rules:

Rule R1: Each vertex in H_1 keeps for itself charge $k - 2/(k - 1)$ and distributes the rest equally among its neighbors of degree $k - 1$.

Rule R2: If a K_{k-1} -subgraph C contains s $(k - 1)$ -vertices adjacent to a $(k - 1)$ -vertex x outside of C and not in a K_{k-1} , then each of these s vertices gives charge $\frac{k-3}{s(k-1)}$ to x .

Claim 53 *Each vertex in H_1 gives to each neighbor of degree $k - 1$ charge at least $\frac{1}{k-1}$.*

Proof. If $v \in H_1$, then v gives to each neighbor charge at least $\psi(d(v)) := \frac{d(v)-k+2/(k-1)}{d(v)}$. Since $\psi(x)$ is monotonically increasing for $x \geq k$, $\psi(d(v))$ is minimized when $d(v) = k + 1$. Then each neighbor of v of degree $k - 1$ gets charge at least $(1 + 2/(k - 1))/(k + 1) = 1/(k - 1)$. \square

Claim 54 *Each vertex in L_1 has charge at least $k - 2/(k - 1)$.*

Proof. Let $v \in L_1$. By Fact 30 every vertex in $L \supseteq L_1$ is in a cluster and that cluster is unique. Let v be in a cluster C of size t . In Cases 1 and 3 we will consider the situation where v is in a $(k - 1)$ -clique. By Claim 31, if X is a $(k - 1)$ -clique, and $v \in X$ then $T \subset X$. Moreover $|N(v) - X| = 1$. By Claim 38, each vertex in $X - C$ has degree at least $k - 1 + t \geq k + 1$, and therefore if Rule R2 applies to v , then it is applied with $t = s$ and it is applied to v at most once.

Case 1: v is in a $(k - 1)$ -clique X and $t \geq 2$. By Claim 50, this case only applies when $k \geq 7$.

By Claim 38, each vertex in $X - C$ has degree at least $k - 1 + t \geq k + 1$, and therefore $X - C \subseteq H_1$. Furthermore, each vertex in $X - C$ has at least $k - 2 - t$ neighbors with degree at least k (the other vertices of $X - C$). Therefore each vertex $u \in (X - C)$ gives charge at least $\frac{d(u)-k+2/(k-1)}{d(u)-k+2+t}$ to each neighbor of degree $k - 1$. Note that this function increases as $d(u)$ increases, so the charge is minimized when $d(u) = k - 1 + t$. It follows that u gives to v charge at least $\frac{t-1+2/(k-1)}{2t+1}$.

So, v has charge at least $k - 1 + (k - 1 - t)(\frac{t-1+2/(k-1)}{2t+1}) - \frac{k-3}{t(k-1)}$, which we claim is at least $k - 2/(k - 1)$. Let

$$g_1(t) = (k - 1 - t)((t - 1)(k - 1) + 2) - (2t + 1)(k - 3)(1 + \frac{1}{t}).$$

We claim that $g_1(t) \geq 0$, which is equivalent to v having charge at least $k - 2/(k - 1)$. Let

$$\tilde{g}_1(t) = (k - 1 - t)((t - 1)(k - 1) + 2) - (2t + 1)(k - 3)(3/2).$$

Note that $\tilde{g}_1(t) \leq g_1(t)$ when $t \geq 2$, so we need to show that $\tilde{g}_1(t) \geq 0$ on the appropriate domain. Function $\tilde{g}_1(t)$ is quadratic with a negative coefficient at t^2 , so it suffices to check its values at the boundaries. They are

$$\tilde{g}_1(2) = (k - 3)(k - 6.5)$$

and

$$\begin{aligned} 4\tilde{g}_1(\frac{k-1}{2}) &= (k - 1)((k - 3)(k - 1) + 4) - 6k(k - 3) \\ &= k^3 - 11k^2 + 29k - 7 \\ &= (k - 7)(k^2 - 4k + 1). \end{aligned}$$

Each of these values is non-negative when $k \geq 7$.

Case 2: $t \geq 2$ and v is not in a $(k-1)$ -clique. By Claim 40, each neighbor of v outside of C has degree at least $k-1+t \geq k+1$ and is in H_1 . Therefore v has charge at least $k-1+(k-t)(\frac{t-1+2/(k-1)}{k-1+t})$. We define

$$\begin{aligned} g_2(t) &= (k-t)(t-1 + \frac{2}{k-1}) - \frac{k-3}{k-1}(k-1+t) \\ &= t(k-t) - 2(1 - \frac{2}{k-1})(k-1) \\ &= t(k-t) - 2(k-3). \end{aligned}$$

Note that $g_2(t) \geq 0$ is equivalent to v having charge at least $k-2/(k-1)$. The function $g_2(t)$ is quadratic with a negative coefficient at t^2 , so it suffices to check its values at the boundaries. They are

$$g_2(2) = 2(k-2) - 2(k-3) = 2$$

and

$$g_2(k-3) = (k-3)(3) - 2(k-3) = k-3.$$

Each of these values is positive.

Case 3: $t = 1$. By definition of L_1 , v is adjacent to at least one vertex w with degree $k-1$. Because $|C| = t = 1$ and so $C = \{v\}$, we have that $w \notin C$ and so by Fact 30 w is in a different cluster. Recall that by definition w, v in different clusters is equivalent to $N[w] \neq N[v]$. If v is not in a $(k-1)$ -clique X , then by Claim 39 w is in a $(k-1)$ -clique and cluster of size at least 2. In this case v will receive charge $(k-3)/(k-1)$ in total from the cluster containing w using Rule R2 and will not give any charge. Therefore we may assume that v is in a $(k-1)$ -clique X .

By Claim 49, there exists a $Y \subset X$ such that $|Y| \geq \frac{k-1}{2}$ and every vertex in Y has degree at least $k+1$. By Claim 31, every vertex in $X - C = X - \{v\}$ is not in a cluster and therefore by Fact 30 every vertex in $X - \{v\}$ has degree at least k . So each vertex in Y has at least $k-3$ neighbors with degree at least k (the vertices of X besides v and itself). Therefore by Rule R1 each vertex $u \in Y$ donates at least $\frac{d(u)-k+2/(k-1)}{d(u)-k+3}$ charge to each neighbor of degree $k-1$. Note that this function increases as $d(u)$ increases, so the charge is minimized when $d(u) = k+1$. It follows that u gives to v charge at least $\frac{1+2/(k-1)}{4}$, and v has charge at least

$$k-1 + \frac{k-1}{2} \left(\frac{1+2/(k-1)}{4} \right) = k + \frac{k-7}{8},$$

which is at least $k-2/(k-1)$ when $k \geq 6$. \square

We then observe that after discharging,

- a) the charge of each vertex in $H_1 \cup L_1$ is at least $k-2/(k-1)$;
- b) the charges of vertices in H_0 did not decrease;
- c) along every edge from H_1 to L_0 the charge at least $1/(k-1)$ is sent.

Thus by Claim 52, the total charge F of the vertices in $H_0 \cup L_0$ is at least

$$kh + (k-1)\ell + \frac{1}{k-1}(\ell(k-1) - e_0) \geq k(h+\ell) - \frac{1}{k-1}2(h+\ell) = (h+\ell) \left(k - \frac{2}{k-1} \right),$$

and so by a), the total charge of all the vertices of G is at least $n \left(k - \frac{2}{k-1} \right)$. Therefore the degree sum of G is at least $n \left(k - \frac{2}{k-1} \right) = \left(\frac{(k+1)(k-2)}{k-1} \right) n$, i. e., $|E(G)| \geq \left(\frac{(k+1)(k-2)}{2(k-1)} \right) n$. \square

6 Sharpness

First we prove Corollary 7, and then we will construct sparse 3-connected k -critical graphs. As it was pointed out in the introduction, Construction 55 and infinite series of 3-connected sparse 4- and 5-critical graphs are due to Toft [33] (based on [32]).

Proof of Corollary 7. By (1), if we construct an n_0 -vertex k -critical graph for which our lower bound on $f_k(n_0)$ is exact, then the bound on $f_k(n)$ is exact for every n of the form $n_0 + s(k-1)$. So, by Corollary 5, we only need to construct

- a 5-critical 7-vertex graph with $\lceil 15\frac{1}{2} \rceil = 16$ edges,
- a 5-critical 8-vertex graph with $\lceil 17\frac{3}{4} \rceil = 18$ edges,
- a 6-critical 10-vertex graph with $\lceil 27\frac{1}{5} \rceil = 28$ edges,
- a 6-critical 12-vertex graph with $\lceil 32\frac{4}{5} \rceil = 33$ edges, and
- a 7-critical 14-vertex graph with $\lceil 45\frac{1}{3} \rceil = 46$ edges.

These graphs are presented in Figure 1. \square

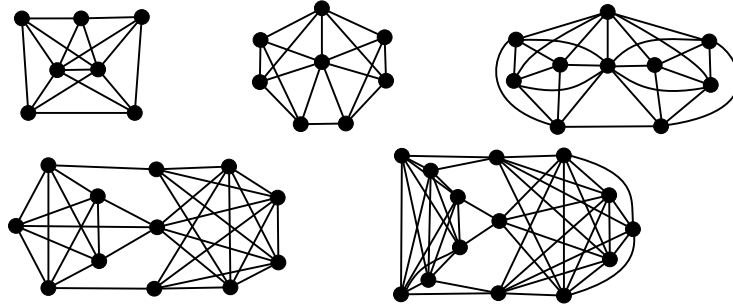


Figure 1: Minimal k -critical graphs.

Construction 55 (Toft [33]) Let G be a k -critical graph, $e = uv \in E(G)$, and $w \in V(G) - \{u, v\}$ be such that for all $(k-1)$ -colorings ϕ of $G - e$, $\phi(w) = \phi(u) = \phi(v)$. Let $S_1 \cup S_2 \cup S_3$ be a partition of the vertex set X of a copy of K_{k-1} such that each S_i is non-empty. We construct G' as $V(G') = V(G) \cup V(X)$ and $E(G') = (E(G) - e) \cup E(X) \cup E'$, where

$$E' = \{ua : a \in S_1\} \cup \{vb : b \in S_2\} \cup \{wc : c \in S_3\}.$$

Claim 56 If G is a 3-connected k -critical graph and G' is created using G and Construction 55, then G' is a 3-connected k -critical graph.

Proof. We will use the names and definitions from Construction 55.

If there exists a $(k-1)$ -coloring ϕ of G' , then all $k-1$ colors must appear on X . Then $\phi(u)$ appears on a vertex in S_2 or S_3 . But then either $\phi(v) \neq \phi(u)$ or $\phi(w) \neq \phi(u)$, which contradicts the assumptions of Construction 55. So $\chi(G') \geq k$.

Suppose there exists an $f \in E(G')$ such that $\chi(G' - f) \geq k$. If $f \in E(G)$, then let ϕ_1 be a $(k-1)$ -coloring of $G - f$. Because $e \in E(G) - f$, $\phi_1(u) \neq \phi_1(v)$, and so ϕ_1 extends easily to $G' - f$. If $f \subset X$, then a $(k-1)$ -coloring of $G - e$ can be extended to $G' - f$, because X can be colored with $k-2$ colors, while $N(X) = \{u, v, w\}$ is colored with 1 color. If $f \in E'$, then a $(k-1)$ -coloring of $G - e$ extends to $G' - f$, because the unique color on $\{u, v, w\}$ can be given to $f \cap X$. Therefore G' is k -critical.

Suppose now that there exists a set S such that $|S| < 3$ and there are nonempty A, B such that $E(A, B) = \emptyset$ and $A \cup B \cup S = V(G')$. Because critical graphs are 2-connected, $|S| = 2$. Because X is a clique, without loss of generality $X \subseteq A \cup S$. By construction, there is no set of size 2 such that $X = A \cup S$, so S also separates $G - e$. Because $\kappa(G) \geq 3$, e has an endpoint in each component of $G - S - e$. But then the components of $G' - S$ are connected with paths through X . \square

The assumptions in Construction 55 are strong. Most edges e in k -critical graphs do not have such a vertex w , and some k -critical graphs do not have any edge-vertex pairs (e, w) that satisfy the assumptions. We will construct an infinite family of sparse graphs with high connectivity, \mathbb{G}_k , that do satisfy the assumptions.

The family is generated for each k by finding a small 3-connected k -critical graph G'_k such that $\rho_k(G'_k) = y_k$. We will describe a subgraph $H'_k \leq G'_k$ with two vertices, u and w , such that in any $(k-1)$ -coloring ϕ' of H'_k , $\phi'(u) = \phi'(w)$. Construction 55 can then be applied to G'_k , using any edge e incident to u that is not in H'_k and not incident to w . Because Construction 55 does not decrease the degree of u , this process can be iterated indefinitely to populate \mathbb{G}_k .

Note that Construction 55 adds the same number of vertices and edges as DHGO-composition with $G_2 = K_k$. Therefore every graph $G \in \mathbb{G}_k$ has $\rho_k(G) = y_k$. Furthermore, G is also k -critical and 3-connected, and therefore not k -Ore. This implies the sharpness of Theorem 6.

All that is left is to find suitable graphs for G'_k and H'_k . Figure 2 illustrates G'_4 and G'_5 . We will need a second construction for larger k .

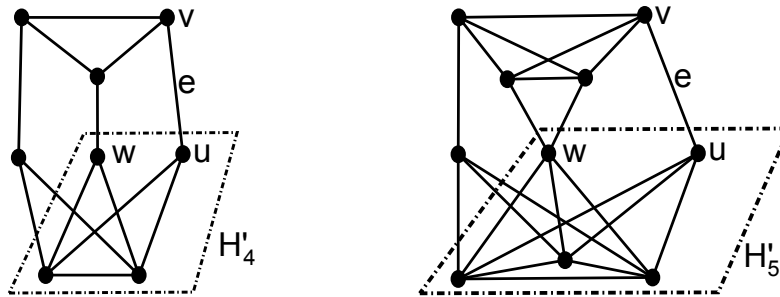


Figure 2: Graphs G'_4 and G'_5 , with substructures labeled for constructing \mathbb{G}_4 and \mathbb{G}_5 .

Construction 57 Fix a t such that $1 \leq t < k/2$. Let

$$V(H_{k,t}) = \{u_1, u_2, \dots, u_{k-1}, v_1, v_2, \dots, v_{k-1}, w\}$$

and

$$E(H_{k,t}) = \{u_i u_j : 1 \leq i < j \leq k-1\} \cup \{v_i v_j : 1 \leq i < j \leq k-1\} \cup \{u_i v_j : i, j \leq t\} \\ \cup \{w u_i : i > t\} \cup \{w v_i : i > t\}.$$

By construction, $H_{k,1}$ is a k -Ore graph, $H_{k,t}$ is k -critical, $\kappa(H_{k,t}) = t+1$, $|V(H_{k,t})| = 2k-1$, and $|E(H_{k,t})| = k(k-1) - 2t + t^2$. Moreover, $\rho_k(H_{k,2}) = y_k$. For $k \geq 6$, we choose $G'_k = H_{k,2}$. We will next find H'_k for $k \geq 6$, which will complete the argument.

Claim 58 *Let $H'_k = H_{k,2} - \{u_1 v_1, u_1 v_2\}$. Then in every $(k-1)$ -coloring ϕ' of H'_k , $\phi'(u_1) = \phi'(w)$.*

Proof. Let ϕ' be a $(k-1)$ -coloring of H'_k . Note that all $(k-1)$ colors appear on $\{u_1, u_2, \dots, u_{k-1}\}$ and appear again on $\{v_1, v_2, \dots, v_{k-1}\}$. Then $\phi'(w)$ appears on a vertex $a \in \{u_1, u_2\}$ and again on a vertex $b \in \{v_1, v_2\}$. So $ab \notin E(G)$, which implies that $a = u_1$. \square

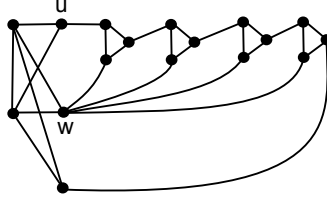


Figure 3: An example of a graph in \mathbb{G}_4 .

7 Algorithm

The proof of Theorem 4 was constructive, and provided an algorithm for $(k-1)$ -coloring of sparse graphs.

Theorem 59 ([26]) *If $k \geq 4$, then every n -vertex graph G with $P_k(G) > k(k-3)$ can be $(k-1)$ -colored in $O(k^{3.5} n^{6.5} \log(n))$ time.*

We present below a polynomial-time algorithm for checking whether a given graph is a k -Ore graph. Together with an analog of the algorithm in Theorem 59 that uses the proof of Theorem 6 instead of Theorem 4, it would yield a polynomial-time algorithm that for every n -vertex graph G with $P_k(G) > y_k$ either finds a $(k-1)$ -coloring of G or finds a subgraph of G that is a k -Ore graph.

Our algorithm to determine whether an n -vertex graph G is k -Ore is simple:

0. If G is K_k , return “yes.”

1. Check whether $n \equiv 1 \pmod{k-1}$ and $|E(G)| = \frac{(k+1)(k-2)|V(G)| - k(k-3)}{2(k-1)}$. If not, then return “no.”
2. Check whether the connectivity of G is exactly 2. If not, then return “no.” Otherwise, choose a separating set $\{x, y\}$.
3. If $G - x - y$ has more than two components or $xy \in E(G)$, then return “no.” Otherwise, let A and B be the vertex sets of the two components of $G - x - y$. If $\{|A| \pmod{k-1}, |B| \pmod{k-1}\} \neq \{k-2, 0\}$, then return “no”. Otherwise, rename A and B so that $|A| \pmod{k-1} = k-2$ and $|B| \pmod{k-1} = 0$.
4. Create graphs $\tilde{G}(x, y)$ and $\check{G}(x, y)$ as defined in Fact 13. Recurse on each of $\tilde{G}(x, y)$ and $\check{G}(x, y)$. If at least one recursion call returns “no,” then return “no.” Otherwise, return “yes.”

The longest procedure in this algorithm is checking whether the connectivity of G is exactly 2 at Step 2, which has complexity $O(kn^3)$ because $|E(G)| \leq kn/2$. And it will be called fewer than $2n/(k-2)$ times. So the overall complexity is at most $O(n^4)$.

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